Geometry & Trigonometry · Unit G20 · Standard

Triangle Truths & the Right-Angle Rule

Standard · MYP 4–5 · 60 practice questions · Criteria A, B, C, D

The twentieth unit of the Geometry & Trigonometry strand, and the tenth of the Standard band. This is a merged unit: it covers both the *triangle properties* row and the *Pythagoras' theorem* row of the official grid, because the first is the natural first half of the second. G02 gave you the angle sum; this unit adds everything else a triangle is obliged to do, and then the one rule that belongs to the right-angled ones alone.

A triangle is the smallest shape you can make with straight lines, and it is the one shape that cannot be pushed out of shape without breaking — which is why roofs, gates, bridges and cell masts are all full of them. This unit is two halves of one story. **First: what is true of every triangle** — which lengths can be a triangle at all, how its angles behave, and what its name tells you. **Then: what is true of the right-angled ones** — the one rule, more than two thousand years old, that turns two sides into the third without a ruler ever touching the page.

By the end of this unit you can

• **Classify** a triangle by its sides and by its angles, and **describe** what the ticks and arcs on a diagram are telling you

• **Apply** the isosceles base-angle property and the angle sum, and **determine** an exterior angle from the two interior angles opposite it

• **Determine** whether three given lengths can be the sides of a triangle, and **explain** what the perpendicular height of a triangle is

• **Calculate** the hypotenuse of a right-angled triangle, giving the answer **exactly** as a simplified surd and to 3 significant figures

• **Calculate** a shorter side by subtraction, and **determine** whether a triangle with three given sides is right-angled

• **Apply** Pythagoras twice to find the space diagonal of a cuboid and the slant height of a pyramid, and **discuss** whether an answer is sensible in its context

The one idea behind all of it

Three lengths either close up into a triangle or they do not, and the test is quick: **the two shorter ones together must be longer than the longest**. Once they do close up, the triangle is fixed — its three angles are decided, and they always add to **180°**. Give one of those angles a special value, **90°**, and something extra becomes true: the side facing the right angle, the **hypotenuse**, is tied to the other two by **$a^2 + b^2 = c^2$**. Square the two you know, add or subtract, take the square root, and you have the one you do not — on paper, in a roof, or across the diagonal of a box.

$a^2 + b^2 = c^2$, where c is the side facing the right angle, and the three angles always add to 180°. Two habits do most of the work: find the hypotenuse before you write anything down, because knowing which side it is decides whether you add or subtract; and keep the answer exact until the very end, because a surd rounded to three figures and then used again in a second stage is an error waiting to grow.

Every length carries its unit and every angle carries its degree sign. A bare 13 is not an answer; 13 cm is. Anything that is not exact is given to 3 significant figures. Round, never truncate.

Where this unit sits

This unit is two grid rows joined at the hip. *Triangle properties* and *Pythagoras' theorem* were listed separately in the blueprint and are built here as one unit, because the first is the natural first half of the second: you cannot ask which side is the hypotenuse until you can name the parts of a triangle, and you cannot check that an answer is possible without the triangle inequality. Unit **G02 Angle Arithmetic** already established that the angles of a triangle add to 180°; that fact is *used* here and is not retaught.

Two things it deliberately does not do. Squares and square roots — including estimating a root between two whole numbers — belong to **N10 Squares, Roots & Powers**, which is already built. **Simplifying a surd**, turning √20 into 2√5, is **Algebra u19 Root and Branch**, also already built; this unit uses simplified surds throughout and links back rather than teaching the simplification again. Neither is a separate page here, and neither is examined as a topic of its own.

What this unit closes and what it opens. It closes the loop left open in **G14 Solid Progress**, where the slant height of a cone and of a pyramid was handed to you as a stated result: here you prove it. It also names the theorem behind **G16 Distance, Midpoint, Gradient** — the distance formula *is* Pythagoras, wearing coordinates, and the right-angled triangle it uses is the one you drop underneath a slanting segment. What it opens is **G21 SOH CAH TOA**: Pythagoras finds a third *side* from two sides, and trigonometry is what you reach for the moment an *angle* is involved.

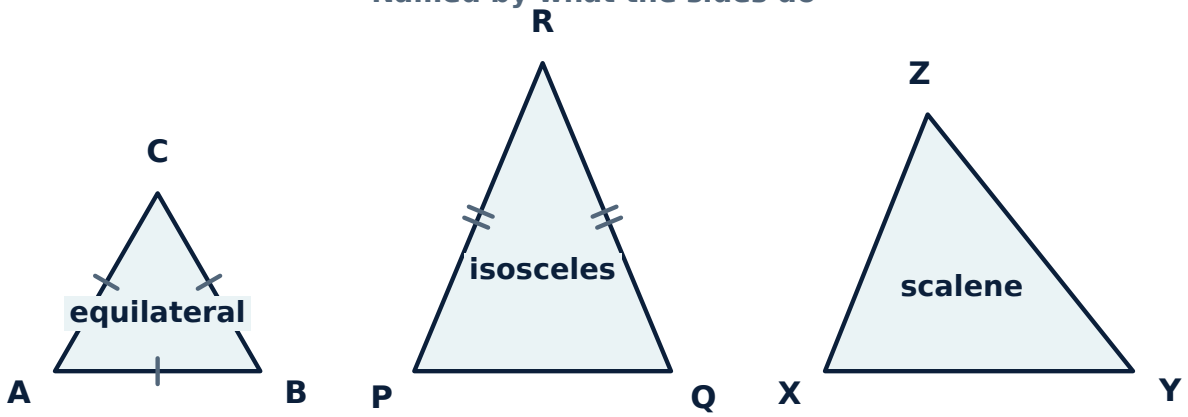
House rules. Every length carries its unit — a bare 13 is not an answer, 13 cm is. Every angle carries its degree sign. An answer that is not a whole number is given **exactly**, as a simplified surd, and then to **3 significant figures** if a decimal is asked for — rounded, never truncated, which is the discipline of **N04 Round It Right**. And in a two-stage calculation the exact value goes into the second stage: rounding in the middle is how the third significant figure gets lost.

Concept 1 · Naming a triangle, and the marks that name it

Classification by sides and by angles, and the isosceles property that links the two

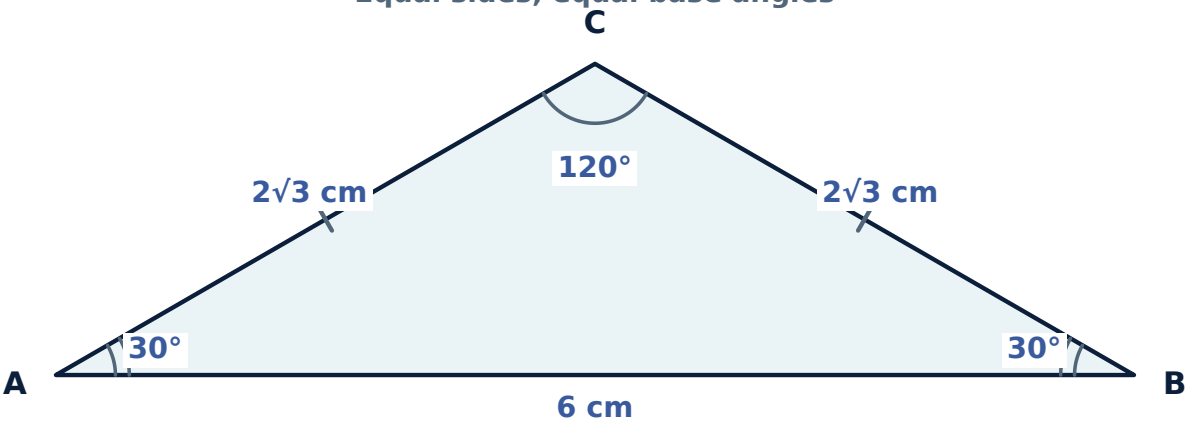
A triangle answers to **two** names, and a question may want either. From its **sides** it is **equilateral** if all three are equal, **isosceles** if exactly two are, and **scalene** if no two are. From its **angles** it is **right-angled** if one is exactly 90°, **obtuse-angled** if one is bigger than that, and **acute-angled** if all three are smaller. The two namings are independent — a triangle can be isosceles and obtuse at the same time — but they are tied together by one fact worth more than any other in this concept: **equal sides face equal angles**. Find two equal sides and you have found two equal angles, and the other way round as well.

Named by what the sides do



The tick marks are not decoration: one tick on all three edges says the three sides are the same length, two ticks on a pair says that pair is equal, and no ticks at all says no two sides are equal. Every tick in this strand is drawn from the measured length, so a tick cannot lie.

Equal sides, equal base angles



The two sides that meet at C are the same length, so the two angles that face them — the base angles at A and at B — are the same size. That is the isosceles property, and it works in both directions: equal sides give equal angles, and equal angles give equal sides.

Worked example 1 · Name it three ways

1	Take the triangle A B C with sides 6 cm, 2√3 cm, 2√3 cm. **Identify** which sides are equal.	**two of them are 2√3 cm**
2	**Classify** it by its sides.	**isosceles**
3	**Calculate** the two base angles — the ones facing the equal sides.	**30° each**
4	**Calculate** the third angle, and check the three add to 180°.	**120°**
5	**Classify** it by its angles.	**obtuse-angled**
6	**Describe** what a diagram would put on it.	**two matching ticks and two matching arcs**

A triangle has two names, not one: a name from its sides and a name from its angles, and a question may want either. The two are linked by the isosceles property — equal sides face equal angles — so finding one pair immediately gives you the other. Notice that the two names here are independent: isosceles says nothing about being obtuse, and obtuse says nothing about being isosceles.

The name	What the sides do	What the angles do	How a diagram marks it
equilateral	all three sides equal	all three angles 60°	one tick on every side
isosceles	exactly two sides equal	the two angles facing those sides are equal	matching ticks on the two equal sides, matching arcs on the two equal angles
scalene	no two sides equal	no two angles equal	no ticks at all
right-angled	the hypotenuse faces the right angle and is the longest	one angle exactly 90°	a small square in the corner

The name	What the sides do	What the angles do	How a diagram marks it
obtuse-angled	the longest side faces the obtuse angle	one angle bigger than 90°, so the other two are small	nothing special — you have to measure or calculate

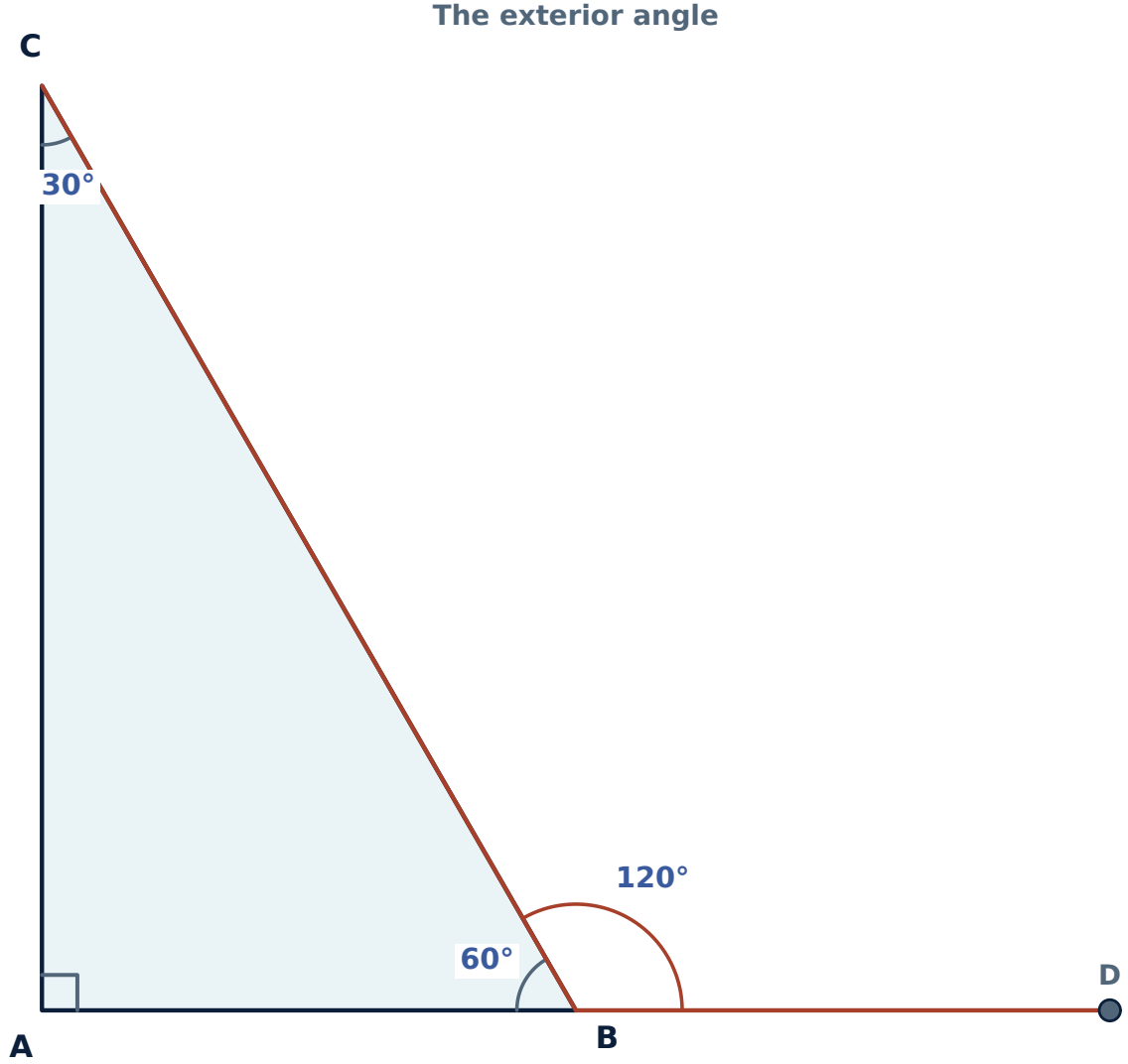
EXAMPLE The triangle with sides 6 cm, 2√3 cm, 2√3 cm has two equal sides, so it is **isosceles**, and the two angles facing those sides are **30°** each. The third angle is **120°**, which is more than a right angle, so the same triangle is also **obtuse-angled**. The triangle with sides 3 cm, 6 cm, 3√3 cm has no two sides equal, so it is **scalene** — and because 9 + 27 = 36 it is also **right-angled**.

Concept 2 · Angles inside, and the angle just outside

The angle sum, the exterior-angle property, and chains that use both

Every triangle spends exactly **180°** on its three angles, which is why knowing two of them always gives you the third. Now carry one side on past a vertex and look at the angle that opens up *outside* the triangle: that is the **exterior angle**, and it is equal to the **two interior angles at the other two vertices added together**. It is not a new fact — the interior and exterior angles at a vertex sit on a straight line and so make 180° between them, exactly like the three interior angles — but it turns a two-step calculation into one step, and in a long chain that saving is where the marks are.

The exterior angle



Carry the side AB on past B to D and the angle DBC that opens up outside the triangle is the exterior angle at B. It is 120°, which is exactly what the two angles at the far end — 90° at A and 30° at C — add up to. That is not a coincidence, and it saves a step every time.

Worked example 2 · Isosceles hunt: one chain, three steps

1	A triangle is isosceles with an apex angle of 40°. **Identify** which two angles must be equal.	**the two base angles**
2	The three angles add to 180°. **Calculate** what is left for the two base angles together.	**140°**
3	**Determine** each base angle.	**70° each**
4	The base is now carried on past one of the base vertices. **Determine** the exterior angle there.	**110°**
5	**Verify** it with the exterior-angle property: add the two interior angles at the other end.	**40° + 70° = 110°**
6	**Explain** which of the two routes you would use in an exam.	**the exterior-angle property — it is one step**

The exterior angle at a vertex equals the two interior angles at the *other* two vertices added together. It is not a new fact — it follows at once from the angle sum, because the interior and exterior angles at one vertex make a straight line — but it turns a two-step calculation into a one-step one, and in a long angle chain that saving is where the marks are.

The triangle	Its three angles	They add to	Its name by angles
the equilateral of the first figure	60° + 60° + 60°	180°	acute-angled
the isosceles A B C above	30° + 30° + 120°	180°	obtuse-angled
the 30-60-90 triangle	90° + 60° + 30°	180°	right-angled
the 3-4-5 triangle	90° + 36.9° + 53.1°	180°	right-angled
the 5-12-13 triangle	90° + 67.4° + 22.6°	180°	right-angled

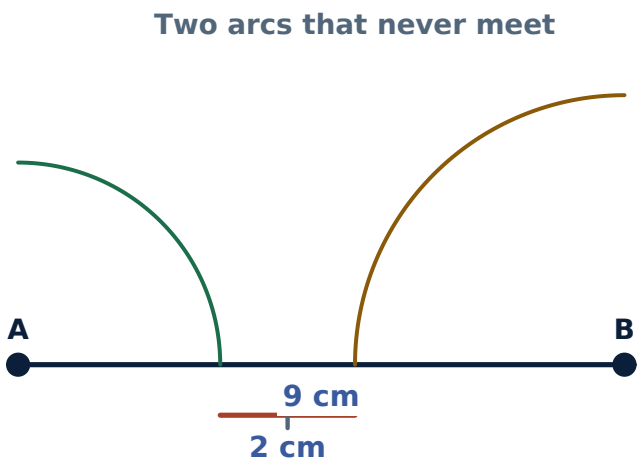
EXAMPLE Two angles of a triangle are 40° and 75°, so the third is 180 − 40 − 75 = **65°**. An isosceles triangle has an apex of 40°, so its base angles share 140° and each is **70°**. And if two interior angles are 43° and

75°, the exterior angle at the third vertex is **118°** — found in one step by adding, or in two by finding the third interior angle first and taking it from 180°.

Concept 3 · Can three lengths even be a triangle?

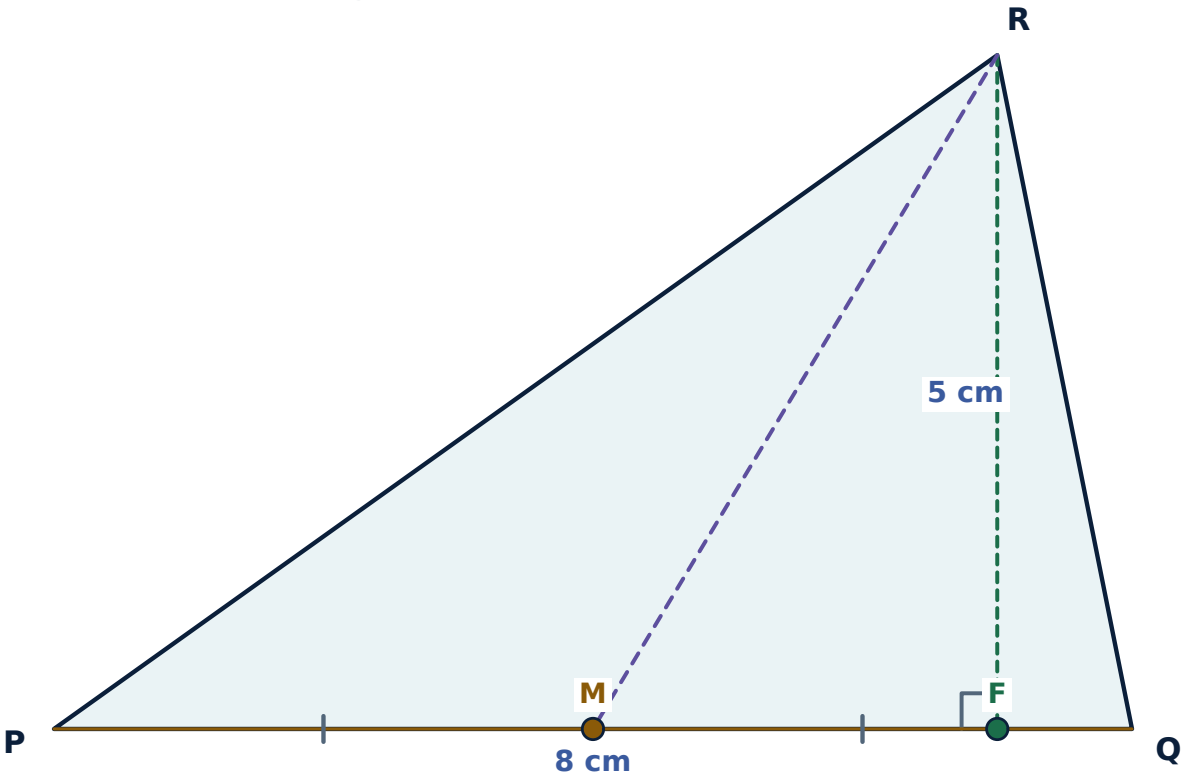
The triangle inequality, the perpendicular height, and what fixes a triangle completely

Not every set of three lengths is a triangle. Lay the longest one down and swing the other two from its ends: if they cross, you have a triangle; if they stop short of each other, you have nothing. The test is **the two shorter lengths added must be greater than the longest** — strictly greater, because if they are exactly equal the three lengths lie flat in a straight line and enclose no area at all. This concept also settles two words that are often confused. The **perpendicular height** of a triangle runs from a vertex to the opposite side at a **right angle**; the **median** runs from a vertex to the **midpoint** of the opposite side. They are different lines, and only the first one is the height an area calculation wants.



Swing an arc of 3 cm from A and one of 4 cm from B and they stop short of each other, with a gap between them. Three lengths only make a triangle when the two shorter ones together are longer than the longest — and 3 + 4 is not more than 9.

The height and the median are different lines



The green dashed line from R meets PQ at a right angle: that is the perpendicular height, and F is its foot. The purple dashed line from R meets PQ at its midpoint M, marked with matching ticks: that is the median. They are two different lines to two different points, and only the first one is the height an area calculation wants.

Worked example 3 · Can these be a triangle?

1	Four sets of lengths. Take 3 cm, 4 cm, 9 cm first and **identify** the longest.	**9 cm**
2	**Calculate** what the other two add to, and compare.	**3 + 4 = 7, which is less than 9**
3	**Determine** whether they make a triangle.	**no — the arcs never meet**
4	Now 5 cm, 12 cm, 13 cm. **Determine** the same thing.	**yes — 5 + 12 = 17, more than 13**
5	Now 2 cm, 2 cm, 4 cm, which is the awkward one.	**no — 2 + 2 = 4 exactly, so it lies flat**
6	And 7 cm, 10 cm, 15 cm.	**yes — 7 + 10 = 17, more than 15**

Only one comparison matters: the two shorter lengths added together against the longest. If the sum is bigger, the triangle exists; if it is smaller, the two short sides cannot reach; and if it is exactly equal, the three lengths lie flat in a straight line and enclose nothing at all. That last case is the one students miss, so read the sign carefully: it must be *greater than*, not *greater than or equal to*.

The three lengths	The longest	The other two added	A triangle?
3, 4, 9 cm	9 cm	7 cm	no
5, 12, 13 cm	13 cm	17 cm	yes
2, 2, 4 cm	4 cm	4 cm	no
7, 10, 15 cm	15 cm	17 cm	yes
6, 6, 6 cm	6 cm	12 cm	yes

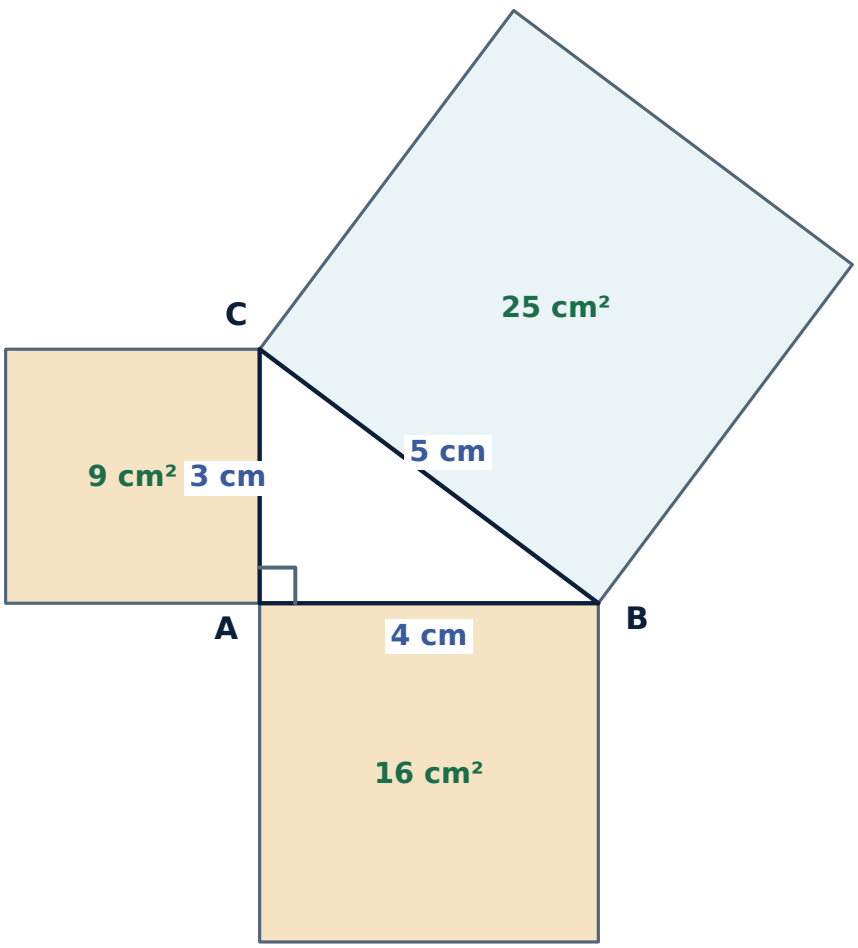
EXAMPLE 3 cm, 4 cm and 9 cm: the two shorter add to 7 cm, which is less than 9 cm, so **no triangle**. 5 cm, 12 cm and 13 cm: 5 + 12 = 17 cm, which beats 13 cm, so **yes**. 2 cm, 2 cm and 4 cm: 2 + 2 = 4 cm exactly, so the three lengths lie flat and there is still **no triangle**. And if two sides are 8 cm and 5 cm, the third must be **more than 3 cm and less than 13 cm**.

Concept 4 · $a^2 + b^2 = c^2$: finding the hypotenuse

Name the side opposite the right angle first, then square, add and root

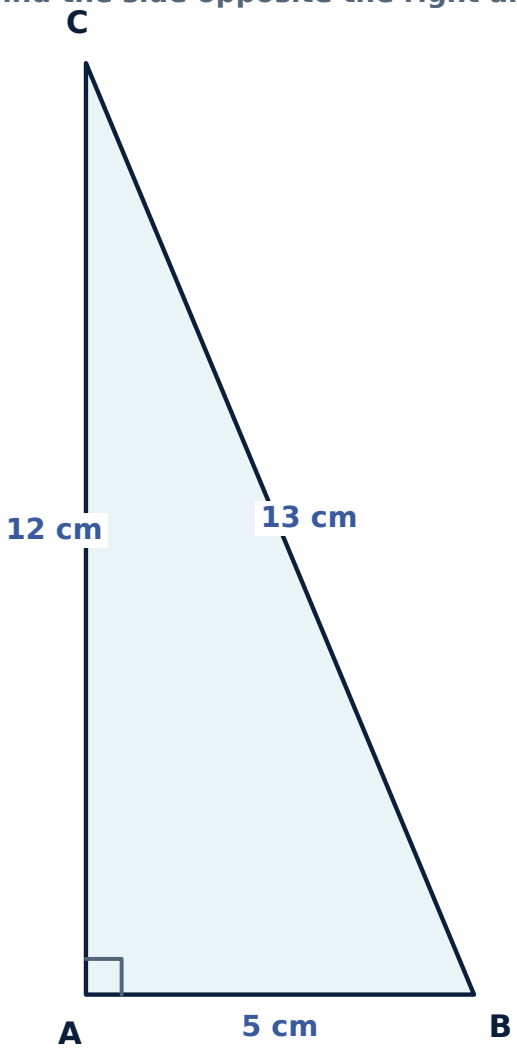
Give a triangle a right angle and something extra becomes true of it. Name the side **facing** the right angle — the **hypotenuse**, always the longest of the three — and call it c, and call the two shorter sides a and b. Then $\mathbf{a^2 + b^2 = c^2}$, always, in every right-angled triangle there has ever been. To find the hypotenuse, square the other two, add, and take the square root. The number under the root is usually not a square number, so the answer is usually a **surd**: leave it as a simplified surd and it is **exact**, write it as a decimal and it is **rounded**, however many figures you give.

A square on every side



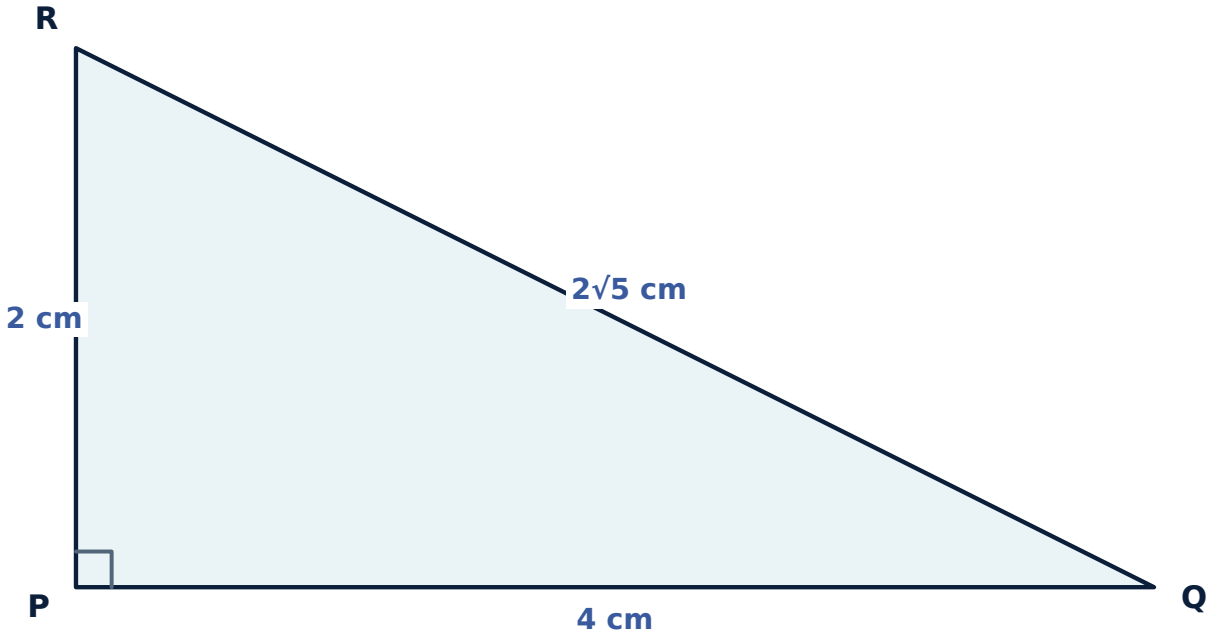
This is the theorem as a picture rather than as a formula. The two small squares, built on the two sides that meet at the right angle, hold 16 cm² and 9 cm² between them. The big square, built on the side opposite the right angle, holds 25 cm² — exactly the other two added together.

Find the side opposite the right angle



Name the hypotenuse before you write anything down: it is the side facing the right angle, BC, and it is always the longest of the three. Squaring the other two and adding gives 25 + 144 = 169, and 169 is a square number, so this answer needs no rounding at all.

An answer that is a surd



Here the two squares add to 20, and 20 is not a square number. The exact length of QR is the simplified surd printed on it; 4.47 cm is that same number rounded to 3 significant figures, and the two are not interchangeable.

Worked example 4 · Find the hypotenuse, exactly and then roundly

1	A right-angled triangle has shorter sides of 2 cm and 4 cm. **Identify** the hypotenuse.	**the side facing the right angle**
2	**Calculate** the two squares and add them.	**20 under the root**
3	20 is not a square number, so the root is not a whole number. Simplify it — that move is Algebra u19's.	**√20 = 2√5**
4	**State** the exact length.	**2√5 cm**
5	**Calculate** the same length to 3 significant figures.	**4.47 cm**
6	**Verify** the answer is sensible: is it longer than 4 cm and shorter than 6 cm?	**yes — 4.47 sits between them**

The exact answer is the surd; the decimal is that number rounded. If a question says *exact*, or *in surd form*, or *leave your answer in the form a√b*, a decimal earns nothing. And the sense check at the end is free: a hypotenuse is always longer than either other side and always shorter than the two of them added, so an answer outside that window is wrong before you check anything else.

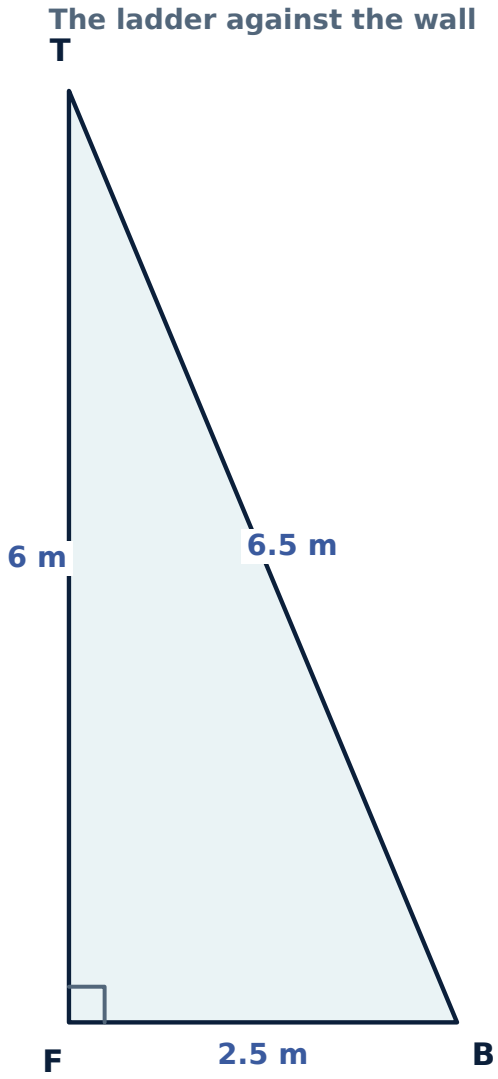
The two shorter sides	Under the root	Exact	To 3 s.f.
3 cm and 4 cm	25	5 cm	5.00 cm
5 cm and 12 cm	169	13 cm	13.0 cm
9 cm and 12 cm	225	15 cm	15.0 cm
2 cm and 4 cm	20	2√5 cm	4.47 cm
1 cm and 1 cm	2	√2 cm	1.41 cm

EXAMPLE Shorter sides of 3 cm and 4 cm give 9 + 16 = 25, so the hypotenuse is **5 cm** and no rounding is needed. Shorter sides of 2 cm and 4 cm give 4 + 16 = 20, and 20 is not a square number, so the exact hypotenuse is **2√5 cm** — which is 4.47 cm to 3 significant figures. A square of side 6 cm has a diagonal of **6√2 cm**, because 36 + 36 = 72.

Concept 5 · Finding a shorter side, and asking whether it is right at all

The subtraction case, and the converse that turns three lengths into a right angle

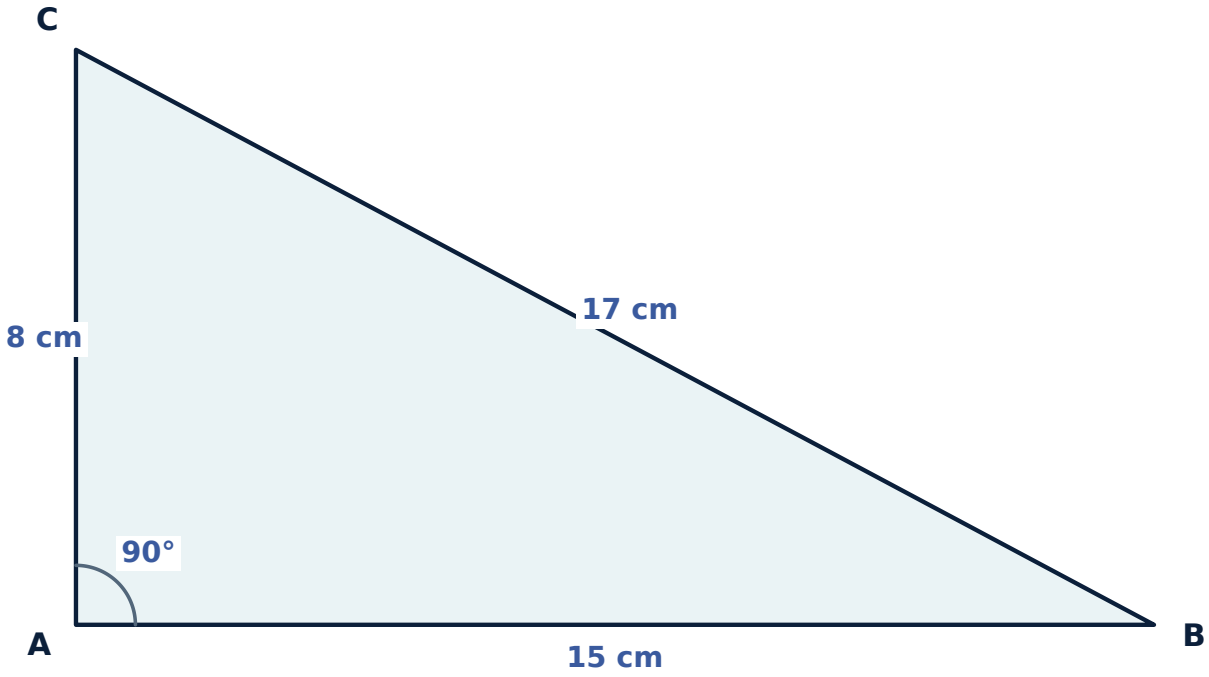
The same rule run the other way finds a **shorter** side. If you know the hypotenuse and one short side, you **subtract**: $c^2 - a^2 = b^2$. Deciding between adding and subtracting is the whole game, and it is decided by one question — is the side I am looking for the one facing the right angle, or not? Then there is the rule read backwards, its **converse**: give me three lengths and no diagram at all, and if the two smaller squares add to the largest square, the triangle *must* be right-angled, with the right angle facing the longest side. If they do not add up, it is not — and which way the comparison falls even tells you whether the biggest angle is obtuse or acute.



The wall is upright and the ground is level, so the ladder is the hypotenuse.

The ladder itself is the hypotenuse, because it faces the right angle where the wall meets the ground. The unknown here is a shorter side, so this one is a subtraction: square the ladder, take away the square of the foot distance, and root what is left.

Test it first, then mark it



Nobody said this triangle was right-angled. Squaring the two shorter sides gives 64 + 225 = 289, and the longest side squared is 17² = 289 as well — so the angle at A really is a right angle. Running the theorem backwards like this is called its converse.

Worked example 5 · The ladder against the wall

1	A 6.5 m ladder leans against an upright wall with its foot 2.5 m from the wall. **Identify** the hypotenuse.	**the ladder itself**
2	**Determine** whether this is an addition or a subtraction.	**a subtraction — the unknown is a shorter side**
3	**Calculate** the two squares.	**$6.5^2 = 42.25$ and $2.5^2 = 6.25$**
4	**Calculate** the difference, which is the square of the height.	**36**
5	**Determine** how far up the wall the ladder reaches.	**6 m**
6	**Discuss** whether the answer is sensible.	**6 m is less than the 6.5 m ladder, as it must be**

Everything in this question turns on one decision made in the first five seconds: the ladder is the hypotenuse, because it faces the right angle where the wall meets the ground. Adding the two given numbers' squares here would give a height of about 7 m — a ladder reaching higher than its own length, which is impossible. The sense check catches that instantly, so always make it.

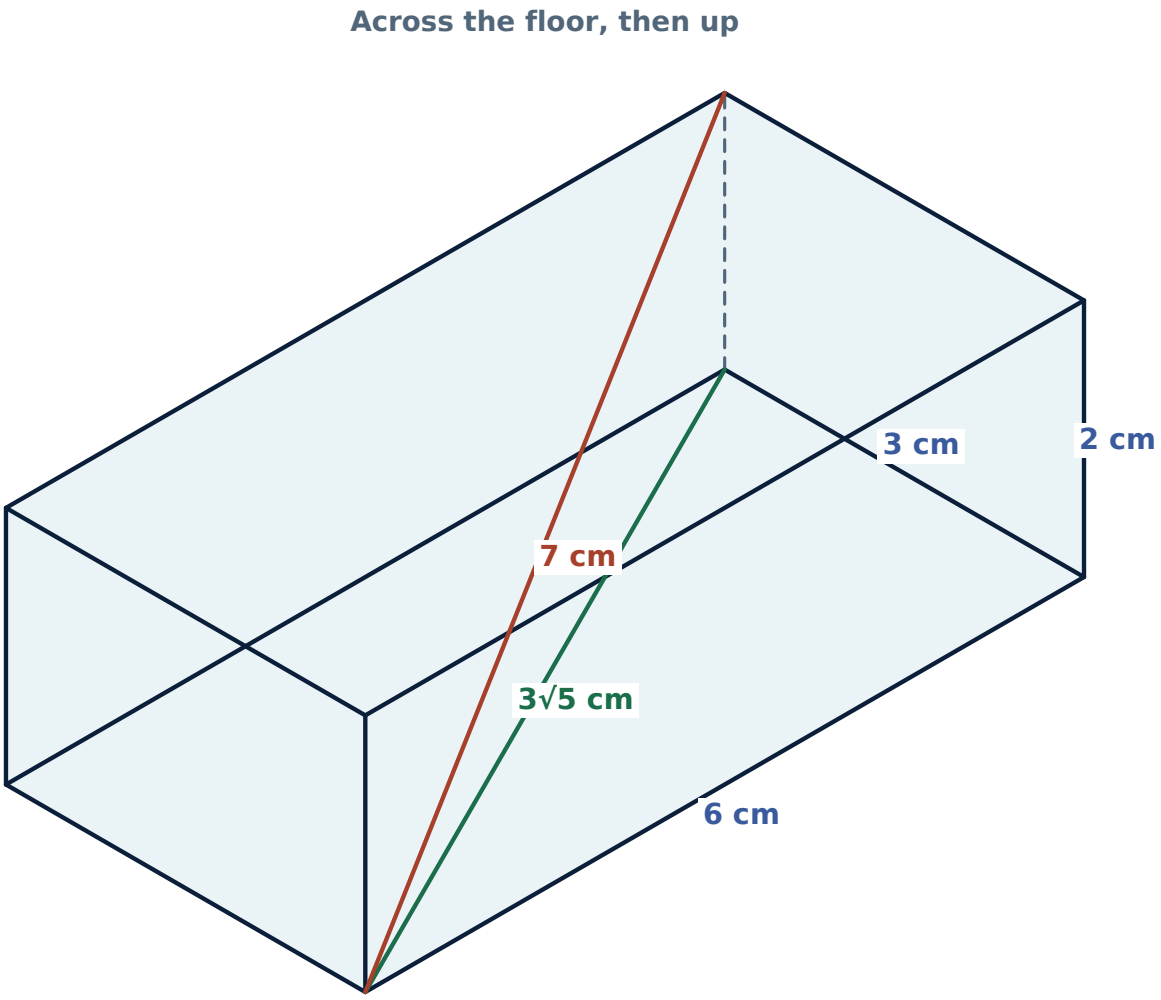
The hypotenuse	The known shorter side	The subtraction	The other shorter side
13 cm	5 cm	144	12 cm
10 cm	6 cm	64	8 cm
25 cm	7 cm	576	24 cm
6.5 cm	2.5 cm	36	6 cm
7 cm	3 cm	40	$2\sqrt{10}$ cm

EXAMPLE A hypotenuse of 13 cm with a short side of 5 cm gives $169 - 25 = 144$, so the other side is **12 cm**. A hypotenuse of 7 cm with a short side of 3 cm gives $49 - 9 = 40$, so the other side is **$2\sqrt{10}$ cm**, or 6.32 cm to 3 significant figures. And for the converse: 8, 15, 17 gives $64 + 225 = 289 = 17^2$, so that triangle **is** right-angled, while 7, 8, 12 gives $49 + 64 = 113$ against 144, so that one **is not**.

Concept 6 · Pythagoras twice over: inside a solid

Across the floor and then up — the space diagonal and the slant height

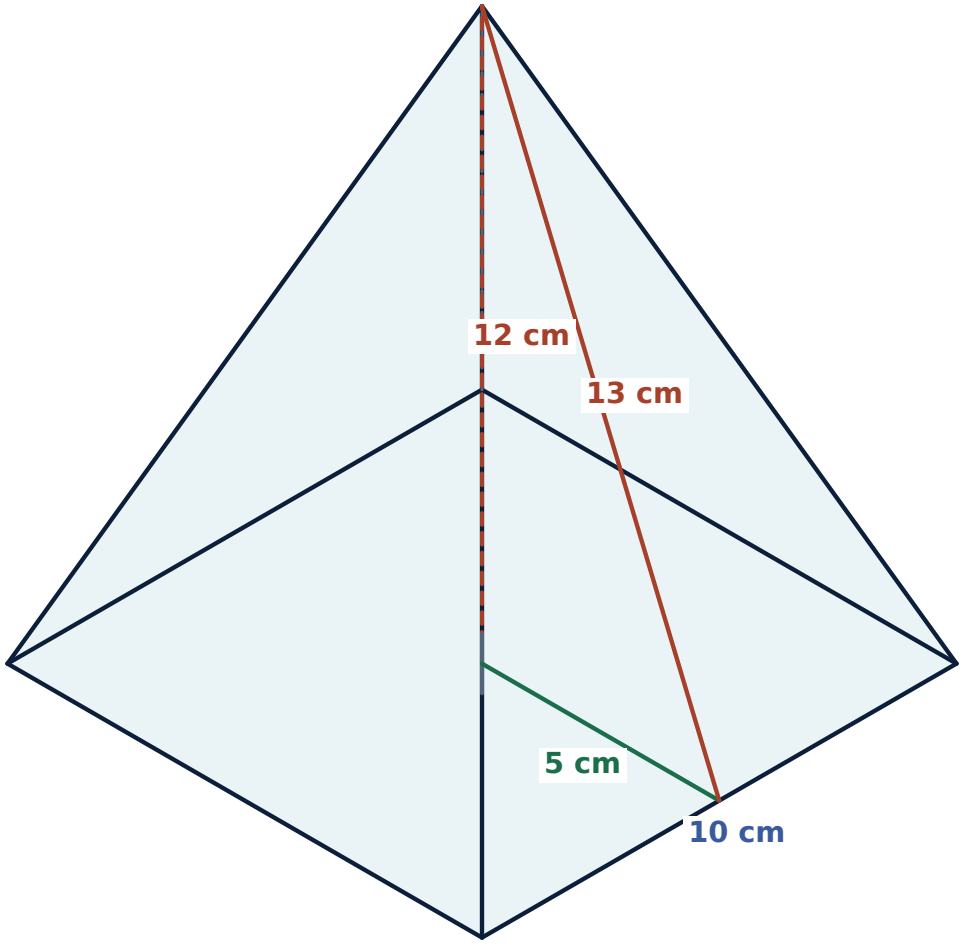
Inside a box, the theorem gets used **twice**. To go from one corner of a cuboid to the opposite one, first cross the floor: that diagonal is the hypotenuse of a right-angled triangle whose sides are the length and the width. Then stand up: the **space diagonal** is the hypotenuse of a second right-angled triangle whose sides are the floor diagonal and the height. The same two-stage move gives the **slant height** of a pyramid, which is exactly the number **G14 Solid Progress** handed you without proof. One warning matters more than all the others here: **feed the exact first answer into the second stage**, surd and all. Round in the middle and the final answer drifts.



Two right-angled triangles, one after the other. The green line runs across the floor of the box and is the hypotenuse of a 6 by 3 triangle. The red line runs from the same corner to the opposite one, and it is the hypotenuse of a triangle whose two sides are

the green line and the height. Feed the exact green answer into the second stage, never a rounded one.

The slant height of a pyramid



The slant height is the red line: from the apex straight down the middle of a triangular face to the middle of a base edge. It is the hypotenuse of a right-angled triangle whose other two sides are the perpendicular height and half the base edge — 12 cm and 5 cm here, which is why the slant height comes out at exactly 13 cm.

Worked example 6 · The space diagonal of a box

1	A box measures 6 cm by 3 cm by 2 cm. **Identify** the two right-angled triangles you will need.	**one on the floor, one standing up**
2	**Calculate** the diagonal across the floor of the box.	**3√5 cm**
3	**Explain** why you should not round that answer yet.	**it is the input to the next stage**
4	The second triangle has that diagonal as one side and the height as the other. **Calculate** what goes under the root.	**49**
5	**Determine** the space diagonal.	**7 cm**
6	**Verify** it in one line: add all three squares.	**36 + 9 + 4 = 49, and √49 = 7**

Pythagoras used twice, and the second use is fed the *exact* output of the first. Here the floor diagonal is 3√5 cm, an irrational number; rounding it to 6.71 cm and squaring that gives 45.0241, not 45, and the final answer drifts off 7. Carrying the surd is not fussiness, it is what keeps the second stage honest — and it is why the one-line check, the sum of all three squares, comes out exact.

Worked example 7 · Show that the slant height really is 13 cm

1	A pyramid has a square base of side 10 cm and a perpendicular height of 12 cm. **Identify** where the slant height runs.	**apex to the middle of a base edge**
2	**Determine** the horizontal distance from the foot of the height to that middle point.	**half the base edge, 5 cm**
3	**Identify** the right-angled triangle those two lines make.	**legs of 12 cm and 5 cm**
4	**Calculate** what goes under the root.	**169**
5	**Show that** the slant height is 13 cm.	**13 cm**
6	**Explain** why the slant height is not the same as the edge from the apex to a corner.	**a corner is further out than a mid-edge**

This closes a loop that **G14 Solid Progress** left open: the slant height of a pyramid or a cone was handed to you there as a stated result so that surface area could be done, and here it is proved. Half the base edge, not the whole base edge, is the number that goes into the triangle — and the slant height is always shorter than the edge to a corner, because a corner is further from the centre than the middle of a side is.

The box	Across the floor	Under the second root	The space diagonal
6 by 3 by 2 cm	3√5 cm	49	7 cm
3 by 4 by 12 cm	5 cm	169	13 cm
2 by 3 by 6 cm	√13 cm	49	7 cm
1 by 2 by 2 cm	√5 cm	9	3 cm
4 by 4 by 7 cm	4√2 cm	81	9 cm

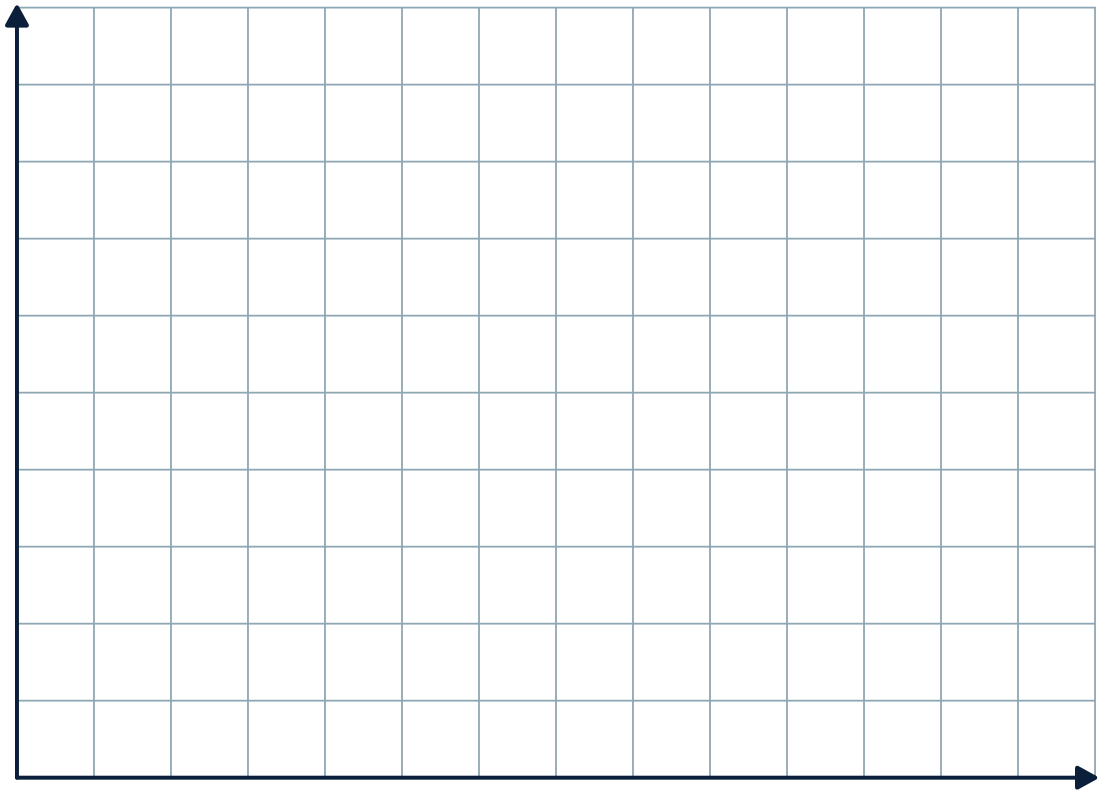
EXAMPLE A box 6 cm by 3 cm by 2 cm has a floor diagonal of **3√5 cm**, and feeding that exact value into the second stage gives 45 + 4 = 49 and a space diagonal of exactly **7 cm**. A cube of side 4 cm has a space diagonal of **4√3 cm**, or 6.93 cm to 3 significant figures. A pyramid on a

square base of 10 cm with a height of 12 cm has a slant height of **13 cm**, because half the base edge is 5 cm and 144 + 25 = 169.

Now construct some for yourself

Construct a triangle with sides of 5 cm, 8 cm and 12 cm on the blank squared paper below: rule the 12 cm side first, then swing a 5 cm arc from one end and an 8 cm arc from the other. Then try 3 cm, 4 cm and 9 cm the same way and watch the arcs fail to meet. Finally **construct** a right-angled triangle with shorter sides of 6 cm and 8 cm along the grid lines, measure its longest side, and check your measurement against 10 cm.

Blank squared paper, for the construction



Number both axes yourself before you construct anything.

Number the axes one square to one centimetre each way. Rule the longest side first, along a grid line, then swing the two other lengths from its ends with a pair of compasses. If the arcs cross, the triangle exists; if they do not, the three lengths have failed the inequality and no amount of redrawing will help.

The right-angle routine

1. Sketch it and mark the right angle before anything else — Even when a diagram is given, redraw it big. Put the square in the right-angle corner, then find the side *facing* that corner and label it c. Everything

else in the calculation follows from getting that one label right.

2. Decide whether you are adding or subtracting — If the unknown is the hypotenuse, you **add** the two squares. If the unknown is one of the shorter sides, you **subtract** the square of the known short side from the square of the hypotenuse. There is no third case, and choosing wrongly is the single commonest error in the topic.

3. Work with the squares, and take the root last — Write the squares down as numbers — 25 and 144, not 5^2 and 12^2 left standing — combine them, and only then take the square root. If the number under the root is not a square number, the exact answer is the simplified surd, and the decimal comes after it, never instead of it.

✓ **Check the answer against the picture and the units** — The hypotenuse must come out longer than either other side and shorter than the two of them added; if it does not, you have added where you should have subtracted. Then check the unit is on the answer and, in a real context, that the number is a size a real thing could be.

The mistakes that cost the most marks

Trap 1 · Adding the two given sides when one of them is already the hypotenuse

WRONG “The ladder is 6.5 m and its foot is 2.5 m out, so the height is $\sqrt{(6.5^2 + 2.5^2)} = 6.96$ m.” The two numbers have been squared and added, out of habit.

RIGHT **Adding is only right when the unknown is the hypotenuse.** Here the ladder *is* the hypotenuse — it faces the right angle between the wall and the ground — so the unknown is a shorter side and the working is a **subtraction**: $42.25 - 6.25 = 36$, and the height is 6 m. The wrong answer, 6.96 m, is a ladder reaching higher up a wall than the ladder is long.

Fix it with one habit, before any arithmetic: **find the right angle, then find the side facing it, then ask whether that side is the one you are looking for**. If it is, add. If it is not, subtract. Writing the letter c on the hypotenuse of your sketch costs two seconds and removes the decision entirely — and the sense check afterwards, that the hypotenuse must be the longest of the three, catches whatever is left.

And **never decide which side is the hypotenuse by looking at the picture**. The figure below is drawn with the vertical direction stretched,

so the side that appears longest on the page is not the longest at all. The hypotenuse is fixed by the right angle and by nothing else: it is the side facing it.

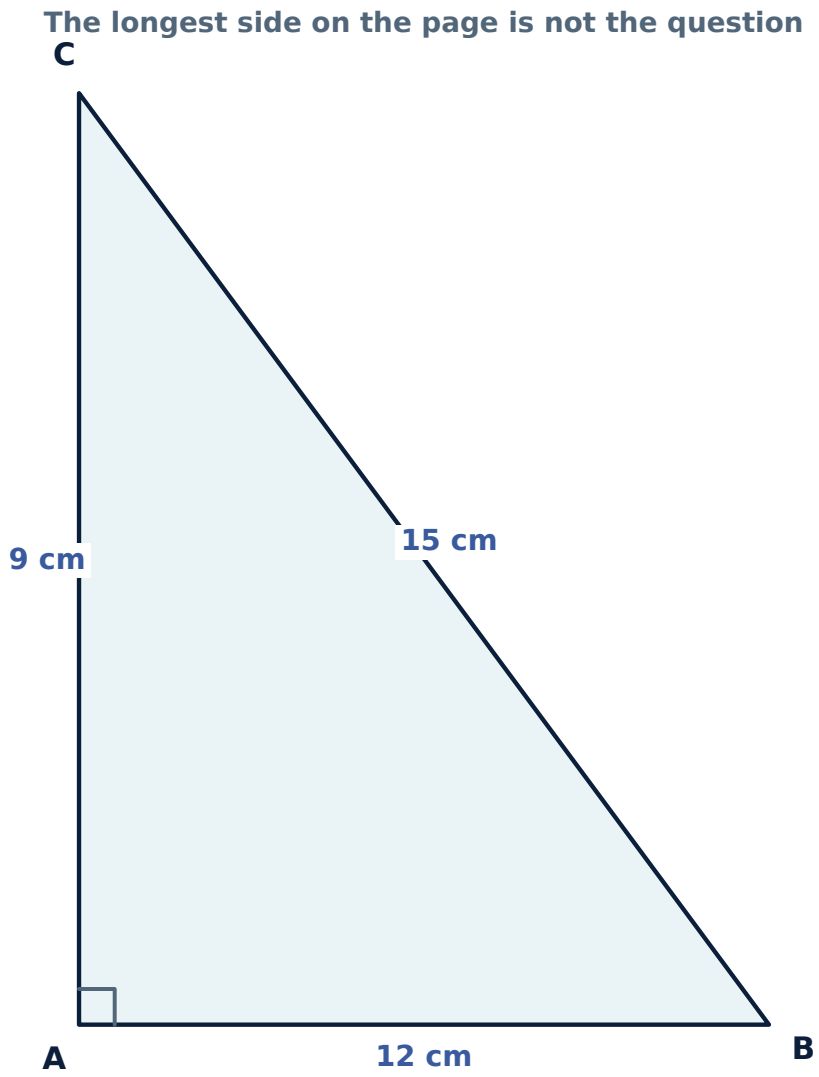


Diagram not to scale

The vertical direction here is stretched, so the drawing is no guide at all to which side is longest. The hypotenuse is decided by the right angle and by nothing else: it is the side facing it, BC, whatever the picture looks like.

Trap 2 · Testing the triangle inequality on only one pair of sides

WRONG “3 cm, 4 cm and 9 cm: well, 4 + 9 = 13, which is bigger than 3, so it is a triangle.” One comparison has been made, and it happens to be the one that was always going to pass.

RIGHT **The comparison that matters is the two shorter sides against the longest.** Any pair that includes the longest side will pass automatically, so testing that pair proves nothing at all. Here 3 + 4 = 7, which is *less* than 9, so the two short sides cannot reach each other and there is no triangle. The picture is two compass arcs that stop short of one another with a 2 cm gap between them.

So sort the three lengths first, or at least spot the longest by eye, and make the one comparison that can fail. Watch the boundary case as well: 2, 2 and 4 give 2 + 2 = 4, exactly equal rather than greater, and the three lengths lie flat along one straight line enclosing no area. The rule is strictly *greater than* — equal is not good enough, and an exam question will test exactly that.

Trap 3 · Writing a rounded decimal where the exact surd was asked for

WRONG “The shorter sides are 2 cm and 4 cm, so the hypotenuse is 4.47 cm.” The number is right, but the question asked for the **exact** length and 4.47 is a rounded one.

RIGHT **Exact means the surd.** The two squares add to 20, so the hypotenuse is √20, which simplifies to 2√5 cm. That is an exact number; 4.47 cm is the same number rounded to 3 significant figures, and the two are not interchangeable. Give the surd whenever the question says *exact*, *in surd form*, or *in the form a√b*.

There is a practical reason as well as a marks reason, and it bites hardest in the three-dimensional questions. A rounded value carried into a second stage drags its error with it: the floor diagonal of a 6 by 3 box is 3√5 cm, and rounding it to 6.71 before squaring gives 45.0241 instead of 45, so the space diagonal comes out at 7.002 instead of exactly 7. Keep the surd all the way through, round once at the very end, and round rather than chop.

Practice · Fluency · 24 questions

P1 **State** the name given to a triangle whose three sides are all the same length.

P2 A triangle has sides of 5 cm, 5 cm and 8 cm. **Classify** it by its sides.

P3 A triangle has sides of 6 cm, 8 cm and 11 cm. **Classify** it by its sides.

P4 The angles of a triangle are 30°, 30° and 120°. **Classify** it by its angles.

P11 Two angles of a triangle are 40° and 75°. **Calculate** the third angle.

P12 A right-angled triangle has an angle of 28°. **Calculate** its third angle.

P13 In triangle A B C the angle at A is 47° and the angle at B is 68°. **Determine** the exterior angle at C.

P14 The exterior angle at one vertex of a triangle is 130°. **Determine** the interior angle at that vertex.

P21 **Determine** whether 3 cm, 4 cm and 9 cm can be the sides of a triangle.

P22 **Determine** whether 5 cm, 12 cm and 13 cm can be the sides of a triangle.

P23 **Determine** whether 2 cm, 2 cm and 4 cm can be the sides of a triangle.

P24 **Determine** whether 7 cm, 10 cm and 15 cm can be the sides of a triangle.

P31 A right-angled triangle has shorter sides of 3 cm and 4 cm. **Calculate** its hypotenuse.

P32 A right-angled triangle has shorter sides of 5 cm and 12 cm. **Calculate** its hypotenuse.

P33 A right-angled triangle has shorter sides of 9 cm and 12 cm. **Calculate** its hypotenuse.

P34 A right-angled triangle has shorter sides of 8 cm and 15 cm. **Calculate** its hypotenuse.

P41 A right-angled triangle has a hypotenuse of 13 cm and one shorter side of 5 cm. **Calculate** the other shorter side.

P42 A right-angled triangle has a hypotenuse of 10 cm and one shorter side of 6 cm. **Calculate** the other shorter side.

P43 A right-angled triangle has a hypotenuse of 25 cm and one shorter side of 7 cm. **Calculate** the other shorter side.

P44 A right-angled triangle has a hypotenuse of 6.5 cm and one shorter side of 2.5 cm. **Calculate** the other shorter side.

P51 A box measures 3 cm by 4 cm by 12 cm. **Calculate** the diagonal across its 3 cm by 4 cm floor.

P52 A box measures 3 cm by 4 cm by 12 cm. **Calculate** its space diagonal.

P53 A box measures 6 cm by 3 cm by 2 cm. **Determine** the exact diagonal across its 6 cm by 3 cm floor.

P54 A box measures 6 cm by 3 cm by 2 cm. **Calculate** its space diagonal.

Practice · Applied · 24 questions

P5 A triangle has sides of 6 cm, 2√3 cm, 2√3 cm. **Classify** it by its sides and **calculate** its two base angles.

P6 A triangle has sides of 3 cm, 6 cm, 3√3 cm. **Classify** it by its sides and by its angles.

P7 Two of the angles of a triangle are 55° and 70°. **Determine** the third angle and **classify** the triangle by its sides and by its angles.

P8 **Explain** why a triangle cannot have two right angles.

P15 An isosceles triangle has an apex angle of 40°. **Calculate** its two base angles.

P16 An isosceles triangle has base angles of 64°. **Calculate** its apex angle.

P17 In an isosceles triangle the exterior angle at one of the base vertices is 150°. **Determine** all three interior angles.

P18 In triangle A B C the exterior angle at C is 118° and the angle at A is 43°. **Determine** the angle at B.

P25 **Determine** whether 9 cm, 12 cm and 20 cm can be the sides of a triangle, and **explain** your reasoning.

P26 Two sides of a triangle are 8 cm and 5 cm. **Determine** the range of possible lengths for the third side.

P27 **Explain** why testing the two shorter sides against the longest is enough, and the other two comparisons need not be made.

P28 A triangle P Q R has PQ = 8 cm and a perpendicular height of 5 cm drawn from R. **Describe** where that height meets PQ, and **explain** how it differs from the median from R.

P35 A right-angled triangle has shorter sides of 2 cm and 4 cm. **Determine** its hypotenuse exactly, and then to 3 significant figures.

P36 A square has sides of 6 cm. **Determine** the exact length of its diagonal.

P37 A gate is 1.6 m wide and 1.2 m high. **Calculate** the length of the diagonal brace that runs corner to corner.

P38 A right-angled triangle has shorter sides of 7 cm and 24 cm. **Calculate** its hypotenuse.

P45 A 6.5 m ladder leans against an upright wall with its foot 2.5 m from the wall. **Calculate** how far up the wall it reaches.

P46 A right-angled triangle has a hypotenuse of 7 cm and one shorter side of 3 cm. **Determine** the other shorter side exactly, and then to 3 significant figures.

P47 **Determine** whether a triangle with sides 8 cm, 15 cm and 17 cm is right-angled.

P48 **Determine** whether a triangle with sides 7 cm, 8 cm and 12 cm is right-angled.

P55 A box measures 2 cm by 3 cm by 6 cm. **Calculate** its space diagonal.

P56 A box measures 1 cm by 2 cm by 2 cm. **Calculate** its space diagonal.

P57 A pyramid has a square base of side 10 cm and a perpendicular height of 12 cm. **Calculate** its slant height.

P58 A pyramid has a square base of side 16 cm and a perpendicular height of 15 cm. **Calculate** its slant height.

Practice · Challenge · 12 questions

P9 An isosceles triangle has an apex angle of x°. **Determine** an expression for each of its base angles.

P10 **Show that** every equilateral triangle is also isosceles and acute-angled.

P19 **Show that** the three exterior angles of any triangle add to 360°.

P20 A triangle has angles of 90°, 60° and 30°. **Show that** the exterior angle at the 60° vertex is equal to the two interior angles opposite it added together.

P29 The sides of a triangle are x cm, (x + 3) cm and (x + 7) cm. **Determine** the smallest whole-number value of x for which the triangle exists.

P30 **Explain** which sets of three measurements fix a triangle completely, and **discuss** why knowing all three angles does not.

P39 A right-angled triangle has two shorter sides of $3\sqrt{2}$ cm each. **Show that** its hypotenuse is a whole number of centimetres.

P40 A rectangle measures 24 cm by 10 cm. **Calculate** the length of its diagonal, and **hence determine** the perimeter of the triangle formed by the diagonal and two sides of the rectangle.

P49 A rectangle has a diagonal of 26 cm and one side of 24 cm. **Calculate** its other side, and **hence determine** its area.

P50 **Show that** a triangle with sides 9 cm, 40 cm and 41 cm is right-angled, and **state** which angle is the right angle.

P59 A cube has sides of 4 cm. **Determine** the exact length of its space diagonal.

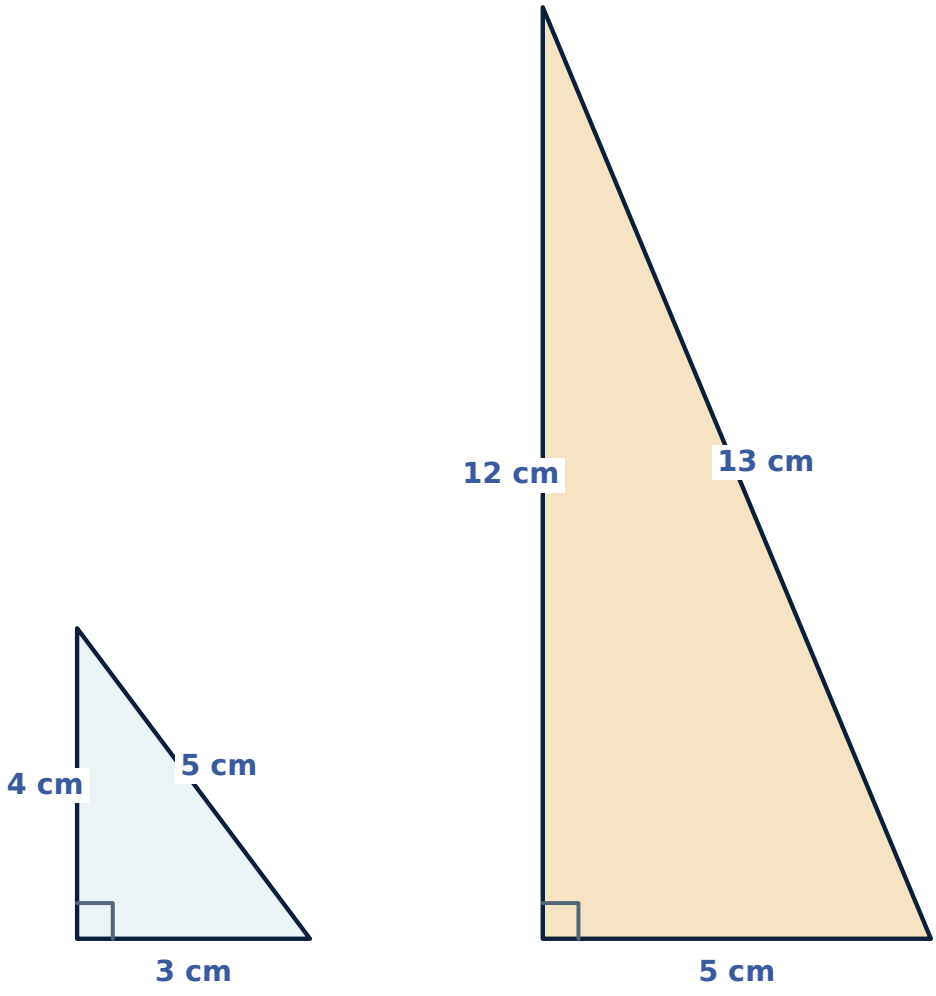
P60 A pyramid has a square base of side 16 cm and a perpendicular height of 14 cm. **Calculate** the length of the edge running from the apex to one corner of the base.

Criterion B · Investigating patterns

The triple factory

Some right-angled triangles come out in whole numbers: 3, 4, 5 is the famous one, and 5, 12, 13 is the next. Sets of three whole numbers that fit $a^2 + b^2 = c^2$ are called **Pythagorean triples**, and hunting for them one at a time is slow work. There is a machine that makes them. Choose any two whole numbers m and n with m bigger than n, and build three lengths from them: $\mathbf{m^2 - n^2}$, $\mathbf{2mn}$ and $\mathbf{m^2 + n^2}$. That is the factory. Your job is to find out what it produces, whether everything it produces really is a right-angled triangle, and which of its products are genuinely new rather than an old one scaled up. It is not a rule you have met, it is in none of the units before this one, and the last part of it has an answer that will surprise you.

Two triples straight off the factory floor



The pair (2, 1) produces 3, 4, 5 and the pair (3, 2) produces 5, 12, 13. Both are right-angled, and neither needed a square root to find: the factory hands you three

whole numbers that fit the theorem exactly.

Part 1 · structured

The first two rows are given so that the column headings are unambiguous: a and b are the two shorter sides in either order, and c is the longest.

(m, n)	a	b	c	$a^2 + b^2$	c^2	Primitive?
(2, 1)	3	4	5	25	25	yes
(3, 2)	5	12	13	169	169	yes
(4, 1)						
(4, 3)						
(5, 2)						
(3, 1)						
(4, 2)						

1. **Predict**, before you calculate anything, whether you think every pair (m, n) will produce a right-angled triangle, and whether you think every Pythagorean triple can be produced this way. Write both predictions down; you will be asked to compare them with what you find.

2. **Complete** the table for the seven pairs given. Work out the three lengths from the pair, then square the two shorter ones and add, then square the longest one, and finally decide whether the triple is *primitive* — that is, whether its three numbers have no common factor bigger than 1. The first two rows are filled in for you.

3. **Identify** the pattern. **Describe** what is always true of the two columns headed $a^2 + b^2$ and c^2, and **explain** what that tells you about every triangle the factory makes.

4. **Determine** a general rule for which pairs (m, n) give a *primitive* triple and which give one that is just a smaller triple scaled up. Look hard at (3, 1) and at (4, 2) before you commit yourself. State the rule as a condition on m and n, not as a list of cases, and use it to **predict** what the pair (5, 4) will produce and whether it will be primitive.

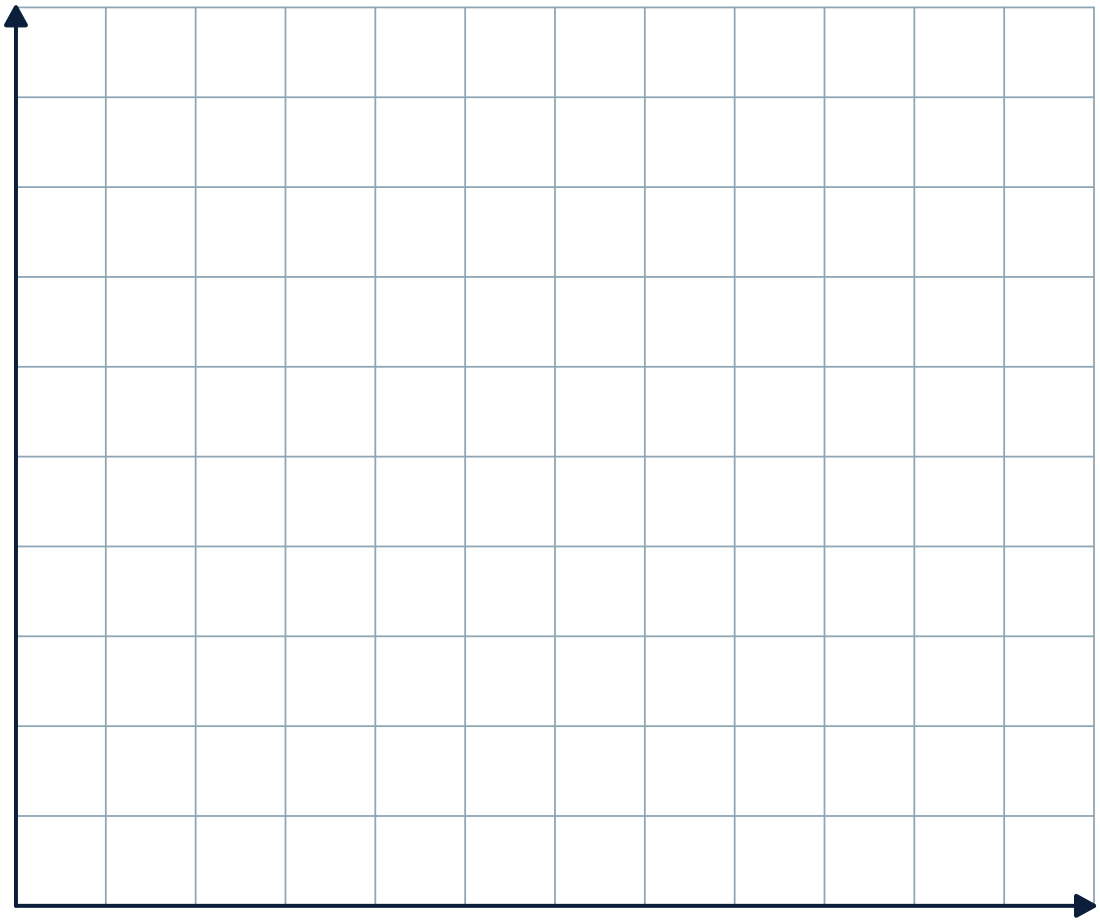
5. **Test and verify** your rules. Generate three more triples from pairs of your own choosing, at least one of which you expect to be primitive and at least one of which you expect not to be. Check every one against $a^2 + b^2 = c^2$, and check your primitivity rule against each. Then **discuss** whether any pair could ever fail to make a triangle at all.

6. **Justify** the factory algebraically. Expand $(m^2 - n^2)^2 + (2mn)^2$ and expand $(m^2 + n^2)^2$, and **explain** why the two expansions must be equal for every m and every n. Then **explain** why that single piece of algebra proves something no amount of table-filling could: that the factory works for numbers nobody has ever tested.

Your own squared paper · for drawing your triples

Draw the two shorter sides of each triple along the grid lines so that the right angle is exact, then measure the third side and check it against the number the factory predicted. A triple that draws correctly is worth more to your write-up than one you only checked with arithmetic.

Your own squared paper



Draw each triple to scale on your own squared paper.

Number the axes yourself, one square to one centimetre each way. Draw the two shorter sides of each triple along the grid lines so that the right angle is exact, then join them and check the third side against the number the factory predicted.

Part 2 · open

Now you choose what to investigate. Take one of these openings or propose your own. You decide which cases to try, how many you need, how to record what you find and how you will test it.

1. Which triples are missing? **Investigate** whether every primitive triple can be produced by the factory. Try to find the pair (m, n) that produces

20, 21, 29 and the pair that produces 33, 56, 65, and **discuss** what your search suggests about the completeness of the machine.

2. Odd and even. **Investigate** the parity of the three numbers in a primitive triple — how many are odd, how many are even, and whether that ever changes. **Determine** a rule and **explain** it from the forms $m^2 - n^2$, $2mn$ and $m^2 + n^2$.

3. Nearly square. **Investigate** the triples for which the two shorter sides differ by exactly 1 — 3, 4, 5 and 20, 21, 29 are the first two. Find the next one or two, **determine** how the pairs (m, n) that produce them are related to each other, and **discuss** whether the list ever stops.

Whichever you choose, take it through the same ladder: **predict**, **identify** the pattern, **determine** a general rule, **test and verify** it on a case you have not already worked out, and **justify** why it must be true.

Marked out of 8 against Criterion B. Completing the table earns the lower bands; the primitivity rule, the test on pairs you were not given, and the algebraic justification are what reach levels 5–8. Part 2 must show your own choice of method.

Remember, you should communicate in an organized and coherent manner.

Criterion D · Applying maths in real-life contexts

How long is a rafter?

A family in Mogoditshane is putting a roof on a house they are building themselves. It is a simple gable roof: the walls go up, a ridge beam runs along the middle, and pairs of straight rafters lean from the wall plates up to the ridge. The builder has written out a sheet of measurements, and it **records a great deal more than you need**. Two questions have to be answered from it. **How long must each rafter be, from wall plate to ridge?** And **the rafters have already been ordered at 4.2 m — does that length leave a sensible overhang past the wall, or has the family bought the wrong timber?** Everything you need is a length somebody could measure with a tape, and the shape that answers both questions is a right-angled triangle you will have to find for yourself in the drawing.

The builder’s sheet (illustrative figures, written for this task)

• **The house:** a family house being built in Mogoditshane, roofed with a simple gable roof

• **The roof shape:** two flat slopes meeting at a ridge that runs along the middle of the house

• **The span:** the distance from one wall plate straight across to the other is 7.2 m

• **The rise:** the ridge sits 1.5 m above the level of the wall plates

• **The rafters:** 24 of them have been ordered, each 4.2 m long

• **The spacing:** the rafters are to be set 0.6 m apart along the ridge

• **The length of the house:** 12 m from end to end along the ridge

• **The walls:** the wall plates sit 2.7 m above the floor

• **The ceiling:** a flat ceiling is to be fixed at 2.4 m inside

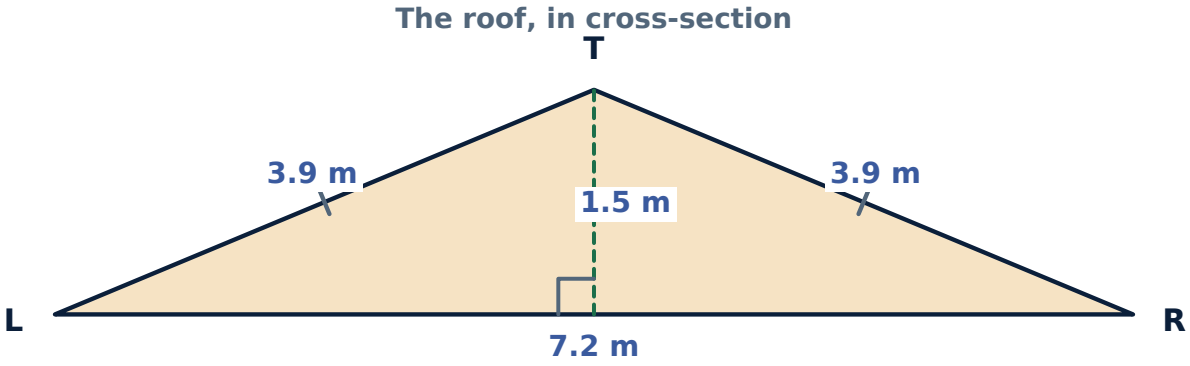
• **The plot:** the plot is 900 m²

• **The overhang:** the family want the roof to project past the wall on each side so that rain runs clear of the wall face

• **How it was measured:** with a 5 m steel tape, every reading written down to the nearest centimetre

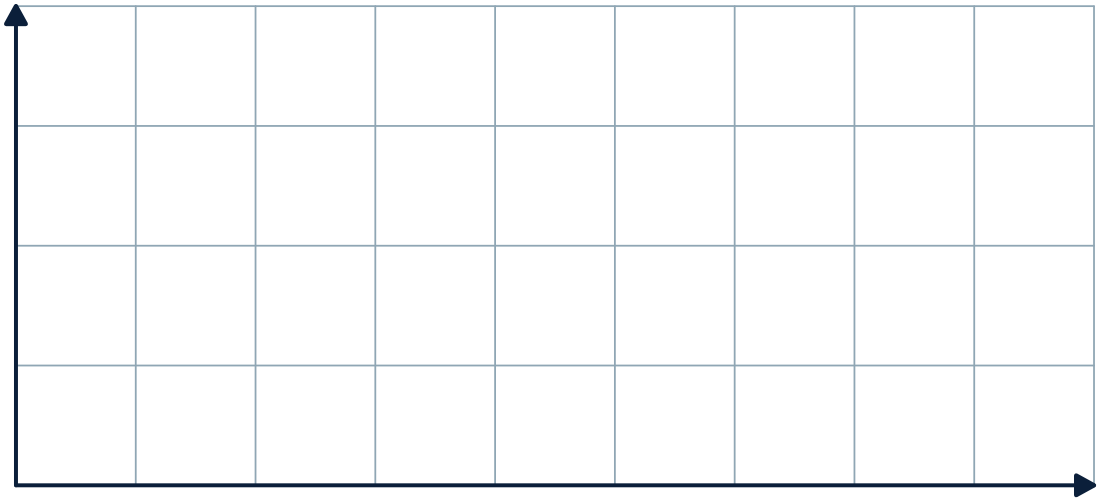
What was measured	The measurement
Span, wall plate to wall plate	7.2 m
Ridge rise above the wall plates	1.5 m
Rafters ordered	24 lengths of 4.2 m
Rafter spacing along the ridge	0.6 m
Length of the house along the ridge	12 m
Wall height, floor to wall plate	2.7 m
Ceiling height inside	2.4 m
Plot area	900 m²

These figures are illustrative. The house, the builder's sheet and the rafter order are an invented illustrative example written for the task. It is not taken from any business, price list or publication, and no organisation, person or source is named or implied. What is authentic is the shape of the problem: a roof really is two right-angled triangles back to back, rafters really are ordered longer than the calculated length to give an overhang, and a tape really is read to the nearest centimetre.



L and R are the two wall plates and T is the ridge. The two rafters are equal, so they carry matching ticks. The green dashed line is the rise from wall-plate level to the ridge, and it meets the span at a right angle exactly halfway along — which is what splits the roof into two right-angled triangles.

Blank axes for the roof section



Choose one scale and use it on both axes.

Metres along the bottom and metres up the side. Use the same scale on both, because this is a cross-section of a real roof and two different scales would make every rafter length on it a lie.

Identify which entries on the builder’s sheet are needed to answer the two questions and which are not, and **explain**, item by item, why each of the others cannot change either answer — deal with the rafter spacing, the length of the house, the wall height, the ceiling height and the plot area by name. Then say which single surplus entry you would still want to know about before the first rafter is cut, and why. *(Strand i — identifying the elements of the authentic situation)*

Select the method you will use. **Explain** where the right-angled triangle is in this roof, which of its three sides is the hypotenuse and how you know, and why the number that goes into the calculation is *half* the span rather than the whole of it. Say in advance how you will check that your answer is sensible. *(Strand ii — selecting adequate mathematical strategies)*

Apply them. Sketch the roof in cross-section on the blank axes, marking the span, the rise and the right angle. **Calculate** the length of one rafter from wall plate to ridge, the total length of rafter timber the family will use, and the overhang the ordered length leaves at each end. State the unit on every answer. *(Strand iii — applying the strategies to reach a solution)*

Explain the degree of accuracy of your answers. Every measurement was written down to the nearest centimetre, so each one could be out by up to half a centimetre. **Explain** what that does to a rafter length worked

out from two of them, whether the answer deserves to be written to the centimetre, and why an answer given to four decimal places would be dishonest here. *(Strand iv — explaining the degree of accuracy of the solution)*

Discuss whether the ordered 4.2 m rafters are the right choice. Say what overhang they leave once the rafter reaches the ridge, **discuss** whether that is enough to keep rain off the wall, and **discuss** what else you would want to allow for before cutting — the notch where the rafter sits on the wall plate, the thickness of the ridge beam, and the fact that timber is sold in fixed lengths. Your recommendation may be to keep the order or to change it, but it must follow from your own numbers. *(Strand v — explaining whether the solution makes sense in context)*

Submit: your cross-section sketch with both axes labelled and scaled, your list of which sheet entries matter and which do not, the rafter length, the total rafter timber and the overhang with their units, and written answers for strands i, iv and v. **Marked out of 8** against Criterion D at Standard level, where the command terms are **explain** and **discuss**. Give every length in metres.

Answers

How the P-numbers run. The 60 practice items are numbered P1 to P60 in the order they appear in the lesson — ten per concept, and inside each concept four Fluency, then four Applied, then two Challenge. The same P-number identifies the item in the lesson, in this key and in the teacher pack, so the numbers inside each tier below step in blocks of ten rather than running consecutively: Fluency is P1–P4, P11–P14, P21–P24 and so on; Applied is P5–P8, P15–P18, …; Challenge is P9–P10, P19–P20, … Every length carries its unit and every angle carries its degree sign. An answer that is not a whole number is given exactly, as a simplified surd, with the 3 significant figure value in brackets after it.

Fluency · F-tier (24)

P1	**equilateral**
P2	**isosceles**
P3	**scalene**
P4	**obtuse-angled**
P11	**65°**
P12	**62°**
P13	**115°**
P14	**50°**
P21	**no**
P22	**yes**
P23	**no**
P24	**yes**
P31	**5 cm**
P32	**13 cm**
P33	**15 cm**
P34	**17 cm**

P41	**12 cm**
P42	**8 cm**
P43	**24 cm**
P44	**6 cm**
P51	**5 cm**
P52	**13 cm**
P53	**3√5 cm (6.71 cm to 3 s.f.)**
P54	**7 cm**

Applied — fully worked (24)

P5 isosceles; the two base angles are 30° each

Two of the sides are 2√3 cm and the third is different, so exactly two sides are equal and the triangle is isosceles. The two angles facing the equal sides are therefore equal, and since the apex angle is 120° the two of them share the remaining 60°, which is 30° each. Equal sides face equal angles: that one sentence is the whole isosceles property, and it works in both directions.

P6 scalene, and right-angled

No two of the three sides are equal, so it is scalene. Squaring the two shorter sides gives 9 + 27 = 36, and the longest side squared is $6^2 = 36$ as well, so the converse of Pythagoras' theorem says there is a right angle — and it faces the 6 cm side. A triangle carries two names, one from its sides and one from its angles, and neither name tells you the other.

P7 55°; isosceles and acute-angled

The three angles add to 180°, so the third is 180 − 55 − 70 = 55°. Two of the angles are now equal, so the two sides facing them are equal and the triangle is isosceles. Every angle is less than 90°, so it is acute-angled. This is the isosceles property used backwards: equal angles give equal sides just as surely as equal sides give equal angles.

P8 two right angles already use the whole 180°, leaving nothing for the third angle

The three angles of any triangle add to exactly 180°. Two right angles come to 90 + 90 = 180° between them, so the third angle would have to

be 0° — and an angle of 0° means the two sides lie on top of one another and the shape has no inside. Drawing it makes the same point: two perpendiculars to the same line are parallel, and parallel lines never meet to make a third corner.

P15 70° each

The two base angles are equal because the two sides facing them are equal. The three angles add to 180°, so the two base angles share 180 − 40 = 140° between them, which is 70° each. Always take the apex away first and then halve; halving 180 first is the slip that turns this into 50°.

P16 52°

Both base angles are 64°, so together they use 128° of the 180° available and the apex gets what is left: 180 − 128 = 52°. This is the reverse of P15, and the check is the same in both directions — add your three angles up and see 180°.

P17 30°, 30° and 120°

The interior angle at that base vertex is 180 − 150 = 30°, because an interior angle and its exterior angle make a straight line. The other base angle is equal to it, so it is 30° too, and the apex is 180 − 30 − 30 = 120°. Check with the exterior-angle property instead: the exterior angle 150° should equal the two opposite interior angles, 30 + 120 = 150°. It does.

P18 75°

The exterior angle at C equals the two interior angles at the other two vertices added together, so 118° = 43° + B and B = 118 − 43 = 75°. The long route works too and is worth doing once as a check: the interior angle at C is 180 − 118 = 62°, and then B = 180 − 43 − 62 = 75°. Same answer, one more line.

P25 yes — 9 + 12 = 21 cm, which is greater than 20 cm

The longest length is 20 cm, so that is the one the other two have to beat. Together they come to 9 + 12 = 21 cm, which is greater than 20 cm, so the two shorter sides can reach each other and the triangle closes up. It closes only just: 21 is barely more than 20, so the triangle is a long thin sliver. The other two comparisons, 9 + 20 against 12 and 12 + 20 against 9, both pass automatically and prove nothing.

P26 greater than 3 cm and less than 13 cm

The third side must be short enough for the other two to reach round it, so it must be less than 8 + 5 = 13 cm. It must also be long enough that the 5 cm side is not left stranded: 5 plus the third side must beat 8, so the third side must be more than 8 − 5 = 3 cm. The answer is the open range

between the difference and the sum, and neither end is included — at exactly 3 cm or exactly 13 cm the three lengths lie flat in a line.

P27 any comparison that includes the longest side passes automatically

Take three lengths and call the longest one c. The comparison a + c > b is certainly true, because c on its own is already at least as big as b and a is positive on top of that; the same argument makes b + c > a certainly true. So those two comparisons can never fail and can never tell you anything. Only a + b > c — the two shorter ones against the longest — is capable of failing, so it is the only one worth making.

P28 the height meets PQ at a right angle, at the foot of the perpendicular; the median meets PQ at its midpoint

The perpendicular height from R is the line from R that crosses PQ at 90°, and where it lands is called the foot — it can be anywhere along PQ, and in an obtuse triangle it can even land outside PQ altogether, on the extension. The median from R is a different line: it runs to the *midpoint* of PQ, cutting it into two equal pieces, and it usually does not meet PQ at a right angle. The two lines coincide only when the triangle is isosceles with PQ as its base. The area of a triangle uses the height, never the median.

P35 2√5 cm (4.47 cm to 3 s.f.)

Squaring and adding gives 4 + 16 = 20, so the hypotenuse is √20. Twenty is 4 × 5 and 4 is a square number, so the root simplifies to 2√5 cm — that is the exact answer. To 3 significant figures it is 4.47 cm. Both are correct answers to different questions, and only the first one is exact.

P36 6√2 cm (8.49 cm to 3 s.f.)

The diagonal cuts the square into two right-angled triangles whose shorter sides are both 6 cm, so the diagonal is the hypotenuse of one of them. Squaring and adding gives 36 + 36 = 72, and √72 simplifies to 6√2 cm, or 8.49 cm to 3 significant figures. Every square behaves this way: the diagonal is always the side length multiplied by √2.

P37 2 m

The gate is a rectangle, so the brace is the hypotenuse of a right-angled triangle with shorter sides 1.6 m and 1.2 m. Squaring and adding gives 2.56 + 1.44 = 4, and √4 = 2, so the brace is exactly 2 m. The numbers were chosen to be a 3-4-5 triangle scaled by 0.4, which is why the answer is so clean — and why the brace must be ordered at 2 m, not 2.8 m.

P38 25 cm

Squaring and adding gives 49 + 576 = 625, and 625 is a square number: √625 = 25. The hypotenuse is 25 cm, and no rounding is needed at all. 7, 24, 25 is worth recognising on sight, alongside 3-4-5, 5-12-13 and 8-15-17.

P45 6 m

The ladder faces the right angle between the wall and the ground, so the ladder is the hypotenuse and the unknown height is a shorter side — which makes this a subtraction. Squaring gives 42.25 for the ladder and 6.25 for the foot distance, and 42.25 − 6.25 = 36. The square root of 36 is 6, so the ladder reaches 6 m up the wall. Sense check: 6 m is less than the 6.5 m ladder, as it has to be.

P46 2√10 cm (6.32 cm to 3 s.f.)

Subtracting the squares gives 49 − 9 = 40, so the side is √40. Forty is 4 × 10, so the root simplifies to 2√10 cm, which is 6.32 cm to 3 significant figures. Check the answer is sensible: it is shorter than the 7 cm hypotenuse and longer than the 3 cm side, which is where a second shorter side has to sit.

P47 yes — 64 + 225 = 289, and 17² = 289

Use the converse of the theorem: square the two shorter sides and add, then compare with the square of the longest. Here $8^2 + 15^2 = 64 + 225 = 289$, and $17^2 = 289$ as well. The two are equal, so the triangle is right-angled, and the right angle is the one *facing* the 17 cm side. Always square the longest side on its own; putting it in with the other two is the slip that turns this test into nonsense.

P48 no — 49 + 64 = 113, but 12² = 144

Squaring the two shorter sides and adding gives $7^2 + 8^2 = 49 + 64 = 113$, while the longest side squared is $12^2 = 144$. They are not equal, so the triangle is not right-angled. It is a real triangle — 7 + 8 = 15, which beats 12 — but the angle facing the 12 cm side is bigger than a right angle, because the square of that side is more than the other two squares put together. That is a useful extra: when c^2 is greater, the triangle is obtuse; when it is smaller, the triangle is acute.

P55 7 cm

Stage one, across the floor: $2^2 + 3^2 = 4 + 9 = 13$, so the floor diagonal is √13 cm — keep it as a surd. Stage two, standing up: that diagonal and the 6 cm height are the two shorter sides of a second right-angled triangle, and 13 + 36 = 49. The square root of 49 is 7, so the space diagonal is exactly 7 cm. The one-line check adds all three squares: 4 + 9 + 36 = 49.

P56 3 cm

Across the 1 cm by 2 cm floor first: 1 + 4 = 5, so the floor diagonal is √5 cm. Then upwards with the 2 cm height: 5 + 4 = 9, and √9 = 3, so the space diagonal is 3 cm. Rounding √5 cm to 2.24 first and squaring gives 5.0176, and the final answer drifts off 3 — which is exactly why the surd is carried.

P57 13 cm

The slant height runs from the apex to the middle of a base edge, and the foot of the perpendicular height is at the centre of the base, which is half the base edge away from that middle point: 10 ÷ 2 = 5 cm. So the right-angled triangle has shorter sides of 12 cm and 5 cm, giving 144 + 25 = 169 and a slant height of 13 cm. Half the base edge, not the whole one, is the number that goes in.

P58 17 cm

Half the base edge is 16 ÷ 2 = 8 cm, so the right-angled triangle has shorter sides of 15 cm and 8 cm. Squaring and adding gives 225 + 64 = 289, and √289 = 17, so the slant height is 17 cm. This is the calculation **G14 Solid Progress** needed when it worked out the surface area of a pyramid and was handed the slant height ready-made.

Challenge — fully worked (12)

P9 (180 − x) ÷ 2 degrees each

The two base angles are equal, so call each of them b. The angle sum gives x + b + b = 180, so 2b = 180 − x and b = (180 − x) ÷ 2. Check it on a case you already know: an apex of 40° gives (180 − 40) ÷ 2 = 70°, which is the answer to Worked example 2. Notice the expression also tells you the apex can never reach 180°, because the base angles would then be 0°.

P10 all three angles are 60°, so it has two equal sides and no angle of 90° or more

Equilateral means all three sides are equal, so in particular *two* of them are equal, which is what isosceles asks for — every equilateral triangle is isosceles, though the reverse is not true. For the angles: equal sides face equal angles, so all three angles are the same, and since they add to 180° each one is 180 ÷ 3 = 60°. Every angle is less than 90°, so the triangle is acute-angled, and this is the only shape of triangle whose angles are fixed before you draw it.

P19 each exterior angle is 180° minus its interior angle, and 3 × 180 − 180 = 360°

Call the interior angles a, b and c, so that a + b + c = 180. The exterior angle at each vertex is 180 minus the interior angle there, so the three exterior angles are 180 − a, 180 − b and 180 − c. Adding them gives 540 − (a + b + c) = 540 − 180 = 360°. The argument never used a particular triangle, so it holds for every one — and it is the same 360° you get by walking right round the outside of the shape and finishing facing the way you started.

P20 the exterior angle is 120°, and 90 + 30 = 120°

The interior angle at that vertex is 60°, so the exterior angle beside it is 180 − 60 = 120°, because the two of them lie along a straight line. The two interior angles at the other vertices are 90° and 30°, and 90 + 30 = 120° as well. The reason is the angle sum: the three interior angles use up 180°, and the straight line at the vertex also measures 180°, so what is left over on the line must be exactly what is left over in the triangle.

P29 x = 5

The longest side is x + 7, so the comparison that can fail is x + (x + 3) > x + 7. That simplifies to 2x + 3 > x + 7, then to x > 4, so the smallest whole number that works is x = 5. Check it: the sides are 5 cm, 8 cm and 12 cm, and 5 + 8 = 13, which is greater than 12 — just. Check x = 4 as well and see it fail: 4 + 7 = 11, exactly equal to 11, so the three lengths lie flat and enclose nothing.

P30 three sides, two sides and the angle between them, two angles and a side, or a right angle with the hypotenuse and one side; three angles fix only the shape, not the size

Three sides fix a triangle: build it with compasses and there is only one place the arcs can cross. Two sides with the angle *between* them fix it, because the third corner is then decided. Two angles and any one side fix it, because the third angle follows from the angle sum and the side sets the scale. And in a right-angled triangle the hypotenuse together with one other side fixes it, because Pythagoras delivers the third side. Three angles fix none of it: they decide the *shape* but not the *size*, so all the triangles with angles of 90°, 60° and 30° are enlargements of one another — which is exactly why AAA is a similarity condition in G18 and not a congruence one.

P39 6 cm

Squaring a surd removes the root sign: $(3\sqrt{2})^2 = 9 \times 2 = 18$. Both shorter sides give 18, so the sum is 18 + 18 = 36, and $\sqrt{36} = 6$. The hypotenuse is exactly 6 cm. Two irrational sides have produced a rational hypotenuse, which is a useful reminder that a surd in the working does not mean a surd

in the answer.

P40 26 cm; the perimeter is 60 cm

The diagonal is the hypotenuse of a right-angled triangle whose shorter sides are 24 cm and 10 cm. Squaring and adding gives 576 + 100 = 676, and √676 = 26, so the diagonal is 26 cm. The triangle it makes with the two sides has a perimeter of 24 + 10 + 26 = 60 cm. Follow through on your own diagonal if it was wrong: the second part is worth its own marks.

P49 10 cm; the area is 240 cm²

The diagonal is the hypotenuse of a right-angled triangle whose other two sides are the sides of the rectangle. Subtracting the squares gives 676 − 576 = 100, and √100 = 10, so the other side is 10 cm. The area of the rectangle is then 24 × 10 = 240 cm². Notice the units change between the two parts: a length in centimetres, an area in square centimetres.

P50 81 + 1600 = 1681 = 41², so it is right-angled; the right angle faces the 41 cm side

Square the two shorter sides and add: $9^2 + 40^2 = 81 + 1600 = 1681$. Square the longest side on its own: $41^2 = 1681$. The two agree exactly, so by the converse of Pythagoras' theorem the triangle is right-angled. The right angle is always the one *opposite* the longest side, so it is the angle between the 9 cm and 40 cm sides. 9, 40, 41 is a genuine Pythagorean triple, and the Criterion B task on this page shows where it comes from.

P59 4√3 cm (6.93 cm to 3 s.f.)

Across the floor first: 16 + 16 = 32, so the floor diagonal is 4√2 cm. Then upwards: 32 + 16 = 48, and √48 simplifies to 4√3 cm, or 6.93 cm to 3 significant figures. Every cube behaves this way: the space diagonal is always the side length multiplied by √3, just as the face diagonal is the side multiplied by √2.

P60 18 cm

This time the triangle reaches a corner, not the middle of an edge, so the horizontal distance is half the base *diagonal*. The base diagonal is 16√2 cm because 256 + 256 = 512, so half of it is 8√2 cm — and its square is 128, which is the number that matters. Then 128 + 196 = 324, and √324 = 18, so the edge is 18 cm. Compare it with P58: the corner is further from the centre than the middle of a side is, so this edge is the longer of the two.