Number · Unit N05

Repeating Forever

Foundations · MYP 1-3 · 60 practice questions · Criteria A, B, C, D

Divide 1 by 3 and the calculator never finishes. Divide 1 by 8 and it stops after three digits. This unit shows you why a fraction only ever does one of those two things, how to write the endless ones down in full, and how to turn them straight back into fractions.

By the end of this unit you can

· Tell a terminating decimal from a recurring one, and say why there is no third kind

· Write a recurring decimal in dot notation, with the dots in the right places

· Turn a fraction into its decimal by dividing, and spot the cycle in the remainders

· Turn a recurring decimal back into a fraction with the $\times 10^n$ subtraction method

· Match the power of 10 to the length of the repeating block

· Decide from the denominator alone whether a fraction will terminate

· Explain why $\sqrt{2}$ and π are not fractions at all

The one idea behind all of it

Everything here comes from watching the remainders while you divide. There are only so many remainders available, so one of two things has to happen: a **remainder of 0** and it stops, or a **repeated remainder** and it repeats. Nothing else can happen, because the remainders run out. Dividing by 7 leaves only six possible non-zero remainders, so within six steps either the division ends or a remainder comes round again - and the moment it does, every later line is a copy of an earlier one.

How we write it · the notation convention

A bar over a block of digits means that block repeats for ever. On screen the same thing is shown with a dot above the first and last digit of the block. A bar and a dot pair mean exactly the same thing.

EXAMPLE $0.\overline{3}$ means 0.333… · $0.\overline{27}$ means 0.272727… · $0.1\overline{6}$ means 0.1666… · $0.\overline{142857}$ means 0.142857142857…
In $0.1\overline{6}$ only the 6 is marked - the 1 happens once and never comes back.

House rules. Give every final answer to 3 significant figures unless it is exact, and always give a fraction in its simplest form. Round, never truncate. Money is in Pula and is always written to 2 decimal places.

Concept 1 · Does it stop, or does it repeat?

The only two things a fraction's decimal can do

Turn any fraction into a decimal by dividing, and watch the remainders. If a remainder of 0 ever appears the division is finished and the decimal **terminates**. If it never appears, a remainder must eventually repeat — there are only finitely many to choose from — and from that moment the digits repeat too. That is a **recurring** decimal. There is no third possibility for a fraction.

1 ÷ 4 · IT STOPS

10 ÷ 4 = 2 remainder 2

20 ÷ 4 = 5 remainder 0

OK

remainder 0 - nothing left to divide

1/4 = 0.25

1 ÷ 3 · IT REPEATS

10 ÷ 3 = 3 remainder 1

10 ÷ 3 = 3 remainder 1

10 ÷ 3 = 3 remainder 1

:

remainder 1 - the same line for ever

1/3 = 0.3

Same starting number, two different endings. 1 ÷ 4 reaches a remainder of 0 and finishes; 1 ÷ 3 returns to remainder 1 and is trapped there for ever.

Worked example 1 · Show that 1/3 = $0.\overline{3}$

1	Set out 1 ÷ 3. Three does not go into 1, so start with 0 and a point	**0.**
2	Bring down a zero: 10 ÷ 3 = 3 remainder 1	**digit 3, remainder 1**
3	Bring down another zero: 10 ÷ 3 = 3 remainder 1 again	**digit 3, remainder 1**
4	The remainder is back to 1, which is exactly where the division started	**the step is forced to repeat**
5	So the digit 3 is produced for ever. One repeating digit, so one dot	**1/3 = $0.\overline{3}$**

The proof is not that we got bored of dividing. It is that the remainder came back to 1. Once a remainder repeats, every later line of the division is a copy of an earlier one, so the digits have no choice but to repeat.

Practice · 10 questions

1.1	**Write down** 1/2 as a decimal.	Fluency
1.2	**Write down** 1/4 as a decimal.	Fluency
1.3	**Write down** 1/5 as a decimal.	Fluency
1.4	**Write down** 1/8 as a decimal.	Fluency
1.5	**Write down** 1/3 as a decimal, using dot notation.	Applied
1.6	**Write down** 1/6 as a decimal, using dot notation.	Applied
1.7	**Write down** 2/9 as a decimal, using dot notation.	Applied
1.8	**Write down** 5/12 as a decimal, using dot notation.	Applied
1.9	**Determine** which of 3/8, 5/6, 7/20 and 4/15 give terminating decimals, and **write down** the other two in dot notation.	Challenge
1.10	**Write down** every remainder that can appear when you divide by 7, before the division ends, and **hence determine** the longest repeating block a seventh could possibly have.	Challenge

Concept 2 · Dot notation

Writing forever without writing forever

A dot above a digit means that digit repeats for ever. When a whole block repeats, put a dot on the first digit of the block and a dot on the last digit of the block, and nothing in between. Digits written before the dots happen once and never come back.

THE PERIOD RULER · 1/7

1/7 = 0.142857 = 0.142857 142857 142857 …

0.	**1**	**4**	**2**	**8**	**5**	**7**	**1**	**4**	**2**	**8**	**5**	**7**	**1**	**4**	**2**	**8**	**5**	**7**
	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18

one period = 6 digits repeat 2 repeat 3

Six digits, then the same six, then the same six again. That is why two dots (or one bar) can replace an infinite amount of writing.

Practice · 10 questions

2.1	**Write down** 0.444… in dot notation.	Fluency
2.2	**Write down** 0.818181… in dot notation.	Fluency
2.3	**Write down** the first six decimal places of $0.\overline{2}$.	Fluency
2.4	**Write down** the first six decimal places of $0.\overline{45}$.	Fluency
2.5	**Write down** 0.5111… in dot notation.	Applied
2.6	**Write down** the first eight decimal places of $0.1\overline{6}$.	Applied
2.7	**Write down** 0.123123123… in dot notation.	Applied
2.8	**Determine** which is larger, $0.\overline{4}$ or 0.45.	Applied
2.9	**Determine** which is larger, $0.1\overline{6}$ or $0.\overline{16}$.	Challenge
2.10	**Write down** the 100th decimal digit of $0.\overline{123}$.	Challenge

Concept 3 · Fraction to decimal, by dividing

Follow the remainders and the cycle appears

Set out the division as usual and keep adding zeros after the point. Write down each remainder as you go. The first remainder that appears twice is the start of the cycle, and everything from there onwards repeats in the same order for ever.

THE DIVISION

$$7 \overline{)\,1.0000000} = 0.1428571\ldots$$

remainder left after each step

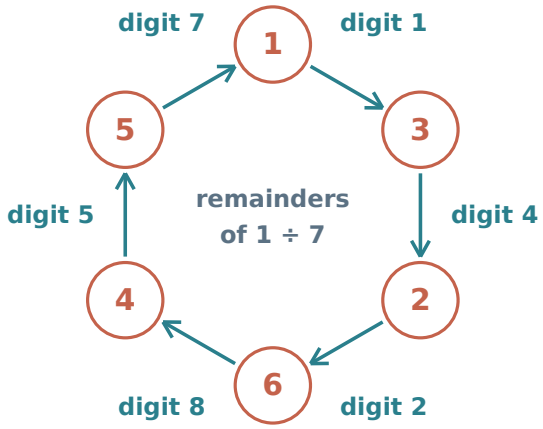
3 2 6 4 5 1

the remainder 1 is back
so every line from here is a copy

1/7 = 0.142857

six digits, then the whole run again

THE REMAINDER CYCLE



six remainders, used once each, then round again

Left: the division, with each remainder written under the digit it produced. Right: the same six remainders drawn as a closed ring. Diagram not to scale.

Practice · 10 questions

3.1	**Divide** to write 1/3 as a decimal.	Fluency
3.2	**Divide** to write 1/9 as a decimal.	Fluency
3.3	**Divide** to write 2/3 as a decimal.	Fluency
3.4	**Divide** to write 1/11 as a decimal.	Fluency
3.5	**Divide** to write 1/7 as a decimal.	Applied
3.6	**Divide** to write 3/11 as a decimal.	Applied
3.7	**Divide** to write 5/6 as a decimal.	Applied
3.8	**Divide** to write 7/12 as a decimal.	Applied

3.9	**Divide** to write 2/7 as a decimal, and **describe** how its repeating block compares with the block for 1/7.	Challenge
3.10	The remainders for 1 ÷ 7 run 1, 3, 2, 6, 4, 5 and then back to 1. **Use** that cycle to write 4/7 as a decimal without dividing again.	Challenge

Concept 4 · Recurring to fraction · one repeating digit

The ×10 subtraction method, with the scaffold given

Give the number a name, x. Multiply by the power of 10 that shifts the decimal point past exactly one block. The two numbers then have identical tails, so subtracting wipes the tail out completely and leaves whole numbers. Solve, then simplify.

The scaffold · use it every time

1. Give the number a name: let x = the recurring decimal.

2. Count the digits in the repeating block. Call it k.

3. Multiply both sides by 10^k - that is 10, 100, 1000 and so on.

4. Subtract the smaller line from the larger. The tails are identical, so they cancel exactly.

5. Solve for x, simplify the fraction, then divide it back out to check.

Worked example 2 · Write $0.\overline{7}$ as a fraction

1	Give the number a name	**$x = 0.\overline{7} = 0.777\ldots$**
2	The block is 1 digit long, so multiply by $10^1 = 10$	**$10x = 7.777\ldots$**
3	Line the two up and subtract. The tails are identical, so they cancel exactly	**$10x - x = 7.777\ldots - 0.777\ldots$**
4	The left side is 9x and the right side is a whole number	**$9x = 7$**
5	Divide by 9, then check nothing simplifies	**$x = 7/9$**
6	Check by dividing back: 7 ÷ 9	**$0.\overline{7}$** ✓

The subtraction is doing all the work. Both tails are the same forever, so they vanish exactly — nothing is rounded, nothing is approximated, and the answer is exact.

Practice · 10 questions

4.1	**Write** $0.\overline{5}$ as a fraction in its simplest form.	Fluency
4.2	**Write** $0.\overline{8}$ as a fraction in its simplest form.	Fluency
4.3	**Write** $0.\overline{1}$ as a fraction in its simplest form.	Fluency
4.4	**Write** $0.\overline{6}$ as a fraction in its simplest form.	Fluency
4.5	**Write** $0.\overline{7}$ as a fraction in its simplest form.	Applied
4.6	**Write** $0.\overline{4}$ as a fraction in its simplest form.	Applied
4.7	**Write** $0.\overline{3}$ as a fraction in its simplest form.	Applied
4.8	**Write** $0.\overline{9}$ as a fraction in its simplest form.	Applied
4.9	**Write** $0.1\overline{6}$ as a fraction in its simplest form.	Challenge
4.10	**Write** $0.58\overline{3}$ as a fraction in its simplest form.	Challenge

Concept 5 · Recurring to fraction · longer blocks

Match the power of 10 to the length of the block

The power of 10 is decided by the block, not by the number of digits you can see. A block of 1 digit needs ×10, a block of 2 digits needs ×100, a block of 3 needs ×1000. If digits sit outside the block, shift past those first and subtract the two lines whose tails match.

Block length	Multiply by	The subtraction leaves
1 digit	×10	9x
2 digits	×100	99x
3 digits	×1000	999x
k digits	$\times 10^k$	a string of k nines times x

If digits sit outside the block, shift past them first and subtract the two lines whose **tails match** - for $0.1\overline{6}$ that is 100x − 10x, not 100x − x.

Worked example 3 · Write $0.\overline{27}$ as a fraction

1	Give the number a name	**x = $0.\overline{27}$ = 0.272727…**
2	The block is 2 digits long, so multiply by $10^2 = 100$	**100x = 27.272727…**
3	Subtract — the tails match, so they cancel	**100x − x = 27**
4	The left side is 99x	**99x = 27**
5	Divide by 99	**x = 27/99**
6	Simplify: 27 and 99 both divide by 9	**x = 3/11**
7	Check by dividing back: 3 ÷ 11	**$0.\overline{27}$** ✓

27/99 was correct but not finished. In this topic more marks are lost on the last line than on the first — always simplify, then divide back to check.

Practice · 10 questions

5.1 **Write** $0.\overline{12}$ as a fraction in its simplest form. Fluency

5.2	**Write** $0.\overline{45}$ as a fraction in its simplest form.	Fluency
5.3	**Write** $0.\overline{81}$ as a fraction in its simplest form.	Fluency
5.4	**Write** $0.\overline{09}$ as a fraction in its simplest form.	Fluency
5.5	**Write** $0.\overline{27}$ as a fraction in its simplest form.	Applied
5.6	**Write** $0.\overline{36}$ as a fraction in its simplest form.	Applied
5.7	**Write** $0.\overline{123}$ as a fraction in its simplest form.	Applied
5.8	**Write** $0.\overline{142857}$ as a fraction in its simplest form.	Applied
5.9	**Write** $0.1\overline{63}$ as a fraction in its simplest form.	Challenge
5.10	**Write** $2.\overline{72}$ as a fraction in its simplest form.	Challenge

Concept 6 · Which denominators terminate

Simplify, then look at the prime factors

Write the fraction in its **simplest form first**, then split the denominator into prime factors. If the only primes there are 2s and 5s, the decimal terminates. If any other prime survives, the decimal recurs. The reason is that our decimals are built on tenths, and 10 = 2 × 5.

40 = 2 × 2 × 2 × 5

2 2 2 5

only 2s and 5s - allowed

7/40 = 0.175 · it stops

12 = 2 × 2 × 3

2 2 3

a prime that is not 2 or 5

5/12 = 0.416 · it repeats

Break the denominator into primes and the answer is immediate. Every 2 and every 5 clears into a power of 10; a 3 never does.

Worked example 4 · Determine which denominators terminate - a mini-investigation

n	1/n as a decimal	Stops?	Prime factors of n
2	0.5	yes	2
3	$0.\overline{3}$	no	3
4	0.25	yes	2 × 2
5	0.2	yes	5
6	$0.1\overline{6}$	no	2 × 3
7	$0.\overline{142857}$	no	7
8	0.125	yes	2 × 2 × 2
9	$0.\overline{1}$	no	3 × 3
10	0.1	yes	2 × 5
11	$0.\overline{09}$	no	11
12	$0.08\overline{3}$	no	2 × 2 × 3

The pattern has nothing to do with how big the denominator is. It is about which primes are hiding inside it. Our decimals are built on tenths, and $10 = 2 \times 5$, so a denominator made only of 2s and 5s can always be turned into a power of 10 — and then it stops. Any other prime never clears, and the division runs for ever.

Practice · 10 questions

6.1	**Determine** whether 1/8 terminates.	Fluency
6.2	**Determine** whether 1/6 terminates.	Fluency
6.3	**Determine** whether 1/20 terminates.	Fluency
6.4	**Determine** whether 1/15 terminates.	Fluency
6.5	**Determine** whether 7/40 terminates, and **write down** the decimal.	Applied
6.6	**Determine** whether 5/12 terminates, and **write down** the decimal.	Applied
6.7	**Determine** whether 6/15 terminates, and **write down** the decimal.	Applied
6.8	**Determine** whether 9/24 terminates, and **write down** the decimal.	Applied
6.9	**Determine** every whole number n from 2 to 20 for which 1/n terminates.	Challenge
6.10	**Determine** the smallest whole number n greater than 1 for which 1/n stops after exactly three decimal places, and the smallest for which it stops after exactly four.	Challenge

The routine

1. Divide, and write the remainders down — A remainder of 0 ends it — the decimal terminates. A remainder you have seen before starts the cycle.

2. Mark the block with dots — One dot for a single repeating digit; a dot on the first and last digit of a longer block. Digits produced before the cycle stay outside.

3. Going back? Name it x and match the power of 10 — Block of k digits needs $\times 10^k$. Digits outside the block need a shift past those first, so subtract the two lines whose tails match.

4. Subtract, solve, simplify, then divide back — The tails cancel exactly. Finish by simplifying the fraction, then check by dividing it out again — you should land back on the decimal you started with.

The mistakes that cost the most marks

Trap 1 · Multiplying by the wrong power of 10

WRONG $0.\overline{27}$: x = 0.272727…, so 10x = 2.727272… and 10x − x = 2.454545…

RIGHT $0.\overline{27}$: the block is 2 digits, so 100x = 27.272727… and 100x − x = 27

Multiplying by 10 shifted the point past only half a block, so the two tails did not match and the subtraction left $9x = 2.\overline{45}$ — still recurring, and no further on. Count the digits in the block first: 1 digit needs ×10, 2 digits need ×100, 3 digits need ×1000.

Trap 2 · Putting the dot over the wrong digit

WRONG $1/6 = 0.\overline{16}$

RIGHT $1/6 = 0.1\overline{6}$

$0.\overline{16}$ means 0.161616…, which is 16/99, not 1/6. The 1 is produced once, before the cycle of remainders begins, so it sits outside the dots. Only digits that come back round again get a dot.

Trap 3 · Assuming every decimal is rational

WRONG “Every decimal can be written as a fraction, so √2 must be one too.”

RIGHT √2 = 1.41421356… and π = 3.14159265… never stop and never repeat, so neither is a fraction.

Terminating and recurring decimals are exactly the fractions — the rational numbers. A decimal that neither stops nor settles into a repeating block, such as 0.101001000100001… where the run of zeros grows every time, is irrational. Recurring means rational; endless-and-never-repeating means irrational.

Criterion B · Investigating patterns

How long before it comes round again?

Every fraction 1/n either terminates or repeats. When it repeats, the number of digits in the repeating block is called the **period**. 1/3 = $0.\overline{3}$ has period 1. 1/11 = $0.\overline{09}$ has period 2. 1/7 = $0.\overline{142857}$ has period 6. The period is not random and it is not obvious. Your job is to find out what decides it.

Part 1 · structured

n	1/n as a decimal	Terminates?	Period
2	0.5	yes	0
3	$0.\overline{3}$	no	1
4			
5			
6			
7			
8			
9			
10			
11			
12			
13			

1. **Complete** the table for every n from 2 to 13. Write each decimal in dot notation, say whether it terminates, and record the period — the number of digits in the repeating block. Use a period of 0 for a decimal that terminates.

2. **Predict** the period of 1/14 and the period of 1/15 *before* you divide. Write both predictions down, then divide and check whether you were right.

3. **Identify** the patterns in your table. Which values of n terminate? Which give a period of exactly 1? Which give a period of exactly 2? And

what do you notice when you compare each period with the value of n itself?

4. **Determine** a general rule that predicts the period for any n you are handed. Two ideas worth testing: *the period is 1 exactly when n divides 9*, and *the period is 2 exactly when n divides 99 but not 9*. If those hold, what would a period of 3 mean? State your rule so that it works for any n, not only the ones in your table.

5. **Verify** your rule on n = 37 and n = 41, neither of which is in your table. Predict the period from the rule first, then divide and check that the two agree.

6. **Justify** why the period of 1/n can never be as large as n. Your explanation should talk about the remainders in the division, not about the digits.

Part 2 · open

Now choose your own line of enquiry. You decide which values of n to test, how many you need, how to set your results out and what would count as convincing evidence. Take one of these, or propose your own:

1. Which values of n give a period of exactly 1, exactly 2, or exactly 3? What do the numbers 9, 99 and 999 have to do with your answer?

2. What happens for values of n that contain a 2 or a 5 as well, such as 6, 12, 14, 15 and 28? Does the repeating block start straight after the decimal point, or later? How much later?

3. Compare the decimals for 1/7, 2/7, 3/7, 4/7, 5/7 and 6/7. Is it the same block every time? Try the same experiment with elevenths and with thirteenths.

4. For prime values of n, compare the period with n − 1. Which primes reach the largest period they possibly could, and how common are they?

5. If you know the period of 1/a and the period of 1/b, can you predict the period of 1/(a × b)? Test it on pairs such as 3 and 11, 7 and 11, 11 and 13.

Whichever you choose, take it through the same ladder: **predict**, **identify** the pattern, **determine** a general rule, **test and verify** it on values you have not already used, and **justify** why it must hold.

Marked out of 8 against Criterion B. Completing the table earns the lower bands; the general rule, the verification and the justification are what reach levels 5-8, and Part 2 must show your own choice of method.

Remember, you should communicate in an organized and coherent manner.

Criterion D · Applying maths in real-life contexts

Splitting the bill in thebe

Six friends finish a meal in Gaborone and ask for one bill so they can split it equally. Here is the slip they were handed. **It contains more information than you need.**

Qty	Item	Each	Line total
2	Seswaa and pap	P89.00	P178.00
2	Grilled chicken quarter and chips	P76.50	P153.00
1	Beef burger and chips	P82.00	P82.00
1	Vegetable pasta	P74.00	P74.00
6	Soft drink 300 ml	P18.00	P108.00
2	Malva pudding (to share)	P75.00	P150.00
	TOTAL DUE		**P745.00**

· Table 9 · served by Kealeboga · 19:42, Saturday

· Prices include VAT at 14%. VAT content of this bill: **P91.49**

· Loyalty points earned on this visit: 74

· Kitchen closes 21:30 · takeaway boxes P6.50 each (none taken)

· The group travelled by kombi, P5.50 each way per person, paid separately

· Payment method: **cash**

Botswana money is written in pula and thebe, and 100 thebe make one pula, so every amount actually paid has exactly two decimal places. The exact share does not.

i · Identify the relevant elements. Identify the figures on the slip that you need in order to split the bill equally, and **describe** why each of the other figures is not needed for that job.

ii · Select appropriate strategies. Describe, in order and before you calculate anything, the steps you will take. Say which division you will do and what you will do with the answer if it does not stop.

iii · Apply them to reach a solution. Calculate the exact share for each of the six friends and write it in dot notation. **Calculate** what each person actually pays, to the nearest thebe, and **calculate** the total the six of them hand over. Then a seventh friend arrives and offers to pay an equal share of the same bill: **calculate** the exact seventh-share in dot notation and what each of the seven would actually pay.

iv · Accuracy of the solution. Describe how accurate your answers are. The exact share never stops, but money is paid in whole thebe, so something has to be given up. **Describe** how much each person is over- or under-paying, and how much the group is over or under in total. **Describe** what changes if everybody rounds down instead of rounding to the nearest thebe.

v · Does it make sense in this situation. Describe whether your answer works in real life. Somebody has to absorb the leftover thebe — **describe** a fair way for the group to settle it. **Describe** what happens if the smallest coin the group has between them is 5 thebe, and **describe** whether an equal split is even the sensible thing to do when two people ordered a P150 dessert that everybody shared.

Submit: your working plus short written answers for strands i, ii, iv and v. **Marked out of 8** against Criterion D. Strands iv and v carry as much weight as the arithmetic - an exact share written to eight decimal places will lose marks if you cannot say what happens to the leftover thebe.

Answers

Fluency · 24 questions

1.1	**0.5**	**4.1**	**5/9**
1.2	**0.25**	**4.2**	**8/9**
1.3	**0.2**	**4.3**	**1/9**
1.4	**0.125**	**4.4**	**2/3**
2.1	**$0.\overline{4}$**	**5.1**	**4/33**
2.2	**$0.\overline{81}$**	**5.2**	**5/11**
2.3	**0.222222**	**5.3**	**9/11**
2.4	**0.454545**	**5.4**	**1/11**
3.1	**$0.\overline{3}$**	**6.1**	**Yes — 8 = 2 × 2 × 2**
3.2	**$0.\overline{1}$**	**6.2**	**No — 6 = 2 × 3**
3.3	**$0.\overline{6}$**	**6.3**	**Yes — 20 = 2 × 2 × 5**
3.4	**$0.\overline{09}$**	**6.4**	**No — 15 = 3 × 5**

Applied · 24 questions, fully worked

1.5 $0.\overline{3}$

1 ÷ 3: the remainder is 1 at every step, so the digit 3 is produced over and over. The block is one digit long, so one dot sits on the 3.

1.6 $0.1\overline{6}$

1 ÷ 6 gives 0.1 with remainder 4. From then on the remainder is 4 every single time, so 6 repeats. The 1 is produced once and never comes back, so the dot goes on the 6 only.

1.7 $0.\overline{2}$

2 ÷ 9 = 0.2 remainder 2, and the remainder 2 returns immediately, so 2 repeats forever.

1.8 $0.41\overline{6}$

5 ÷ 12 = 0.41 with remainder 8, then 80 ÷ 12 = 6 remainder 8 for ever. The 4 and the 1 happen once; only the 6 repeats.

2.5 $0.5\overline{1}$

Only the 1 repeats. The 5 is produced once, so it stays outside the dot.

2.6 0.16666666

The dot is on the 6 only, so the 1 is written once and the 6 fills every place after it.

2.7 $0.\overline{123}$

The block is 123, three digits long. Dots go on the first digit and the last digit of the block, never on the middle ones.

2.8 0.45

$0.\overline{4}$ = 0.4444… and 0.45 = 0.4500…. Compare place by place: the tenths tie at 4, and in the hundredths 4 < 5. So 0.45 is larger.

3.5 $0.\overline{142857}$

Remainders in order: 1, 3, 2, 6, 4, 5, then 1 again. The digits they produce are 1, 4, 2, 8, 5, 7. The moment remainder 1 returns, the whole run starts over.

3.6 $0.\overline{27}$

30 ÷ 11 = 2 remainder 8, then 80 ÷ 11 = 7 remainder 3, and remainder 3 is where we started, so the pair 27 repeats.

3.7 $0.8\overline{3}$

50 ÷ 6 = 8 remainder 2, then 20 ÷ 6 = 3 remainder 2 for ever. Only the 3 repeats.

3.8 $0.58\overline{3}$

70 ÷ 12 = 5 r 10, 100 ÷ 12 = 8 r 4, 40 ÷ 12 = 3 r 4, and the remainder 4 now sticks. The 5 and the 8 happen once each.

4.5 7/9

Let x = $0.\overline{7}$. The block is 1 digit long, so multiply by 10: 10x = $7.\overline{7}$. Subtract: 10x − x = 7.777… − 0.777… = 7, so 9x = 7 and x = 7/9. 7 and 9 share no factor, so it is already simplest.

4.6 4/9

x = $0.\overline{4}$, 10x = $4.\overline{4}$, subtract to get 9x = 4, so x = 4/9.

4.7 1/3

$x = 0.\overline{3}$, $10x = 3.\overline{3}$, subtract to get $9x = 3$, so $x = 3/9$. Divide top and bottom by 3: $x = 1/3$. Always finish by simplifying.

4.8 1

$x = 0.\overline{9}$, $10x = 9.\overline{9}$, subtract to get $9x = 9$, so $x = 1$. This is not a rounding error and not a trick: 0.999… and 1 are two names for one number.

5.5 3/11

The block is 2 digits long, so multiply by 100, not 10. $x = 0.\overline{27}$, $100x = 27.\overline{27}$. Subtract: $99x = 27$, so $x = 27/99$. Divide top and bottom by 9: $x = 3/11$.

5.6 4/11

$x = 0.\overline{36}$, $100x = 36.\overline{36}$, subtract to get $99x = 36$, so $x = 36/99 = 4/11$.

5.7 41/333

The block is 3 digits long, so multiply by 1000. $x = 0.\overline{123}$, $1000x = 123.\overline{123}$, subtract to get $999x = 123$, so $x = 123/999$. Both divide by 3: $x = 41/333$.

5.8 1/7

The block is 6 digits long, so multiply by 1 000 000. That gives $999999x = 142857$, so $x = 142857/999999$. Since $142857 \times 7 = 999999$, the fraction is exactly 1/7.

6.5 Yes, 0.175

$40 = 2 \times 2 \times 2 \times 5$ — only 2s and 5s, so it stops. $7 \div 40 = 0.175$.

6.6 No, $0.41\overline{6}$

$12 = 2 \times 2 \times 3$. The 3 cannot be cleared, so the decimal must repeat: $0.41\overline{6}$.

6.7 Yes, 0.4

Simplify first: $6/15 = 2/5$. The denominator is now 5, so it stops: $2/5 = 0.4$. Testing 15 before simplifying would have given the wrong answer.

6.8 Yes, 0.375

9/24 simplifies to 3/8, and $8 = 2 \times 2 \times 2$, so it stops: 0.375. Unsimplified, $24 = 2 \times 2 \times 2 \times 3$ looks as though it should repeat.

Challenge · 12 questions, fully worked

1.9 3/8 = 0.375 and 7/20 = 0.35 terminate; 5/6 = $0.8\overline{3}$ and 4/15 = $0.2\overline{6}$ recur

Simplify first — all four are already in lowest terms. 8 = 2 × 2 × 2 and 20 = 2 × 2 × 5 contain only 2s and 5s, so those two stop. 6 = 2 × 3 and 15 = 3 × 5 both carry a 3, so those two repeat.

1.10 remainders 1, 2, 3, 4, 5, 6; longest possible block 6 digits

A remainder must be smaller than 7, and a remainder of 0 would end the division, so only 1, 2, 3, 4, 5 and 6 are available. Once a remainder repeats the digits repeat, so the block can never be longer than 6. 1/7 uses all six, so 6 really is reached.

2.9 $0.1\overline{6}$

$0.1\overline{6}$ = 0.16666… and $0.\overline{16}$ = 0.161616…. The tenths tie at 1 and the hundredths tie at 6; in the thousandths 6 > 1. Moving one dot changes the number.

2.10 1

The block 123 is 3 digits long, and 100 = 3 × 33 + 1, so the 100th digit is the 1st digit of a fresh block, which is 1.

3.9 $0.\overline{285714}$; the same six digits in the same cyclic order, entered at a different point

1/7 gives 142857 and 2/7 gives 285714. Reading 142857 round in a circle and starting at the 2 gives 285714 — the cycle of remainders is the same ring, joined in a different place.

3.10 $0.\overline{571428}$

Join the ring at remainder 4. From 4 the remainders continue 4, 5, 1, 3, 2, 6 and the digits they produce are 5, 7, 1, 4, 2, 8. So 4/7 = $0.\overline{571428}$, the same six digits once more.

4.9 1/6

One digit sits outside the block, so shift past it first. x = $0.1\overline{6}$, 10x = $1.\overline{6}$ and 100x = $16.\overline{6}$. Subtract the pair whose tails match: 100x − 10x = 16.666… − 1.666… = 15, so 90x = 15 and x = 15/90 = 1/6.

4.10 7/12

Two digits sit outside the block. x = $0.58\overline{3}$, 100x = $58.\overline{3}$ and 1000x = $583.\overline{3}$. Subtract: 900x = 583.333… − 58.333… = 525, so x = 525/900 = 7/12.

5.9 9/55

One digit is outside the block and the block is 2 digits long. $x = 0.1\overline{63}$, $10x = 1.\overline{63}$ and $1000x = 163.\overline{63}$. Subtract the matching tails: $990x = 163.6363\ldots - 1.6363\ldots = 162$, so $x = 162/990 = 9/55$.

5.10 30/11

The whole-number part comes along for the ride. $x = 2.\overline{72}$, $100x = 272.\overline{72}$, subtract to get $99x = 270$, so $x = 270/99 = 30/11$. Check: $30/11 = 2.727272\ldots$ ✓

6.9 2, 4, 5, 8, 10, 16, 20

These are exactly the numbers built from 2s and 5s only: 2, $4 = 2^2$, 5, $8 = 2^3$, $10 = 2 \times 5$, $16 = 2^4$, $20 = 2^2 \times 5$. Every other n from 2 to 20 carries a prime other than 2 or 5.

6.10 8 and 16

1/2 = 0.5 and 1/5 = 0.2 stop after 1 place; 1/4 = 0.25 after 2; 1/8 = 0.125 after 3; 1/10 = 0.1 after 1. The next terminating denominator is 16, and 1/16 = 0.0625 stops after 4. For a denominator $2^a \times 5^b$ the number of places is the larger of a and b.