Number · Unit N12

The Number Families

Standard · MYP 4–5 · 60 practice questions · Criteria A, B, C, D · the first Standard-band unit

√16 is not irrational. 0.4161616… is not irrational. π is, and so is √20, and knowing which is which is worth a mark every time it is asked. This unit gives the five families of number their proper names and symbols, shows you why they sit inside one another, and hands you a routine that classifies any number you will meet.

By the end of this unit you can

• Name the five families — $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{Q}'$ and $\mathbb{R}$ — and say what each one holds

• Read and write $\in$, $\notin$ and $\subset$ correctly, and know which joins a number to a set and which joins two sets

• Explain why $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, and why $\mathbb{Q}'$ sits beside $\mathbb{Q}$ rather than inside it

• Show that any terminating or recurring decimal is rational by producing the fraction

• Test a number for irrationality properly — never ends AND never repeats

• Classify any number into every family it belongs to, and answer "find a number such that" questions with a counterexample

The one idea behind all of it

There is only one question, and everything in this unit is an answer to it: **can this number be written as a fraction of two integers?** If it can, it is rational. If it cannot, it is irrational. Every symbol, every oval in the diagram and every trap in this unit is that one question wearing different clothes.

a/b, with a and b integers and b ≠ 0. Can it be written that way? Yes → $\mathbb{Q}$. No → $\mathbb{Q}'$.

Where this unit sits

This unit opens the Standard band of the Number course. It is the vocabulary unit: almost nothing in it is new arithmetic, and almost all of it is new language. That is exactly why it is worth doing carefully — the e-assessment gives away one-mark naming questions, and they are only free if you know the words.

It leans on two earlier units and repeats neither. Turning a recurring decimal into a fraction is N05 *Repeating Forever*, and it is used here as evidence, not re-taught from scratch. Simplifying a square root is N10 *Squares, Roots and Powers*; here the point is what the simplified root tells you about which family the number lives in.

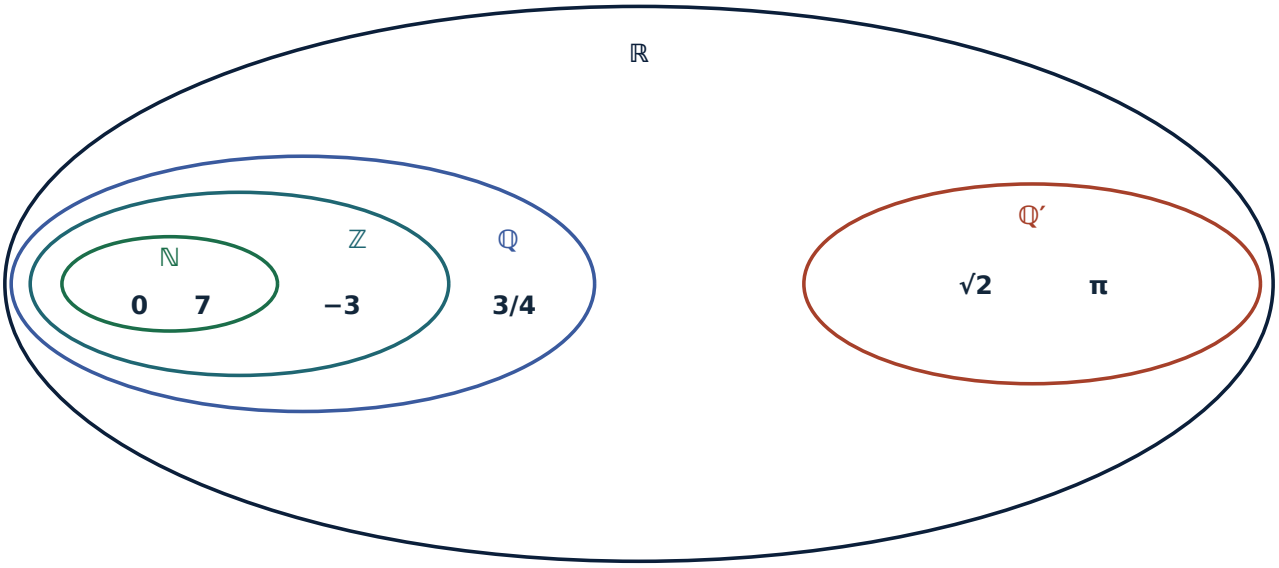
Surds themselves are Algebra Unit 19. This unit only needs to know that √2 is irrational and why. Manipulating surds — simplifying √45, adding like surds, rationalising a denominator — is taught once, in Algebra, and the Standard-band bridge page after N17 links straight to it.

Concept 1 · The five families

ℕ, ℤ, ℚ, ℚ′ and ℝ — what each one holds

Mathematics sorts numbers into five named families, and each one has a symbol you are expected to recognise on sight. Four of them nest, each sitting wholly inside the next; the fifth, the irrational numbers, fills in what is left over.

THE FIVE FAMILIES, AND HOW THEY SIT INSIDE ONE ANOTHER



ℝ = ℚ ∪ ℚ′ — every real number is one or the other, and none is both.

ℕ the positive integers and zero · ℤ the integers · ℚ the rational numbers · ℚ′ the irrational numbers · ℝ the real numbers. Each oval is drawn wholly inside the next, because every member of the smaller family really is a member of the larger one. ℚ′ is drawn beside ℚ, never inside it.

Worked example 1 · Name the smallest family holding each of 7, −4, 2/3 and √5

1	Start at the smallest family and work outwards	ℕ → ℤ → ℚ → ℚ′
2	7 is whole and not negative, so the first family already holds it	**7 ∈ ℕ**
3	−4 is whole but negative, so ℕ is too small	**−4 ∈ ℤ**
4	2/3 is not whole, but it IS a fraction of two integers	**2/3 ∈ ℚ**
5	√5 = 2.236067977… never ends and never repeats	**√5 ∈ ℚ′**

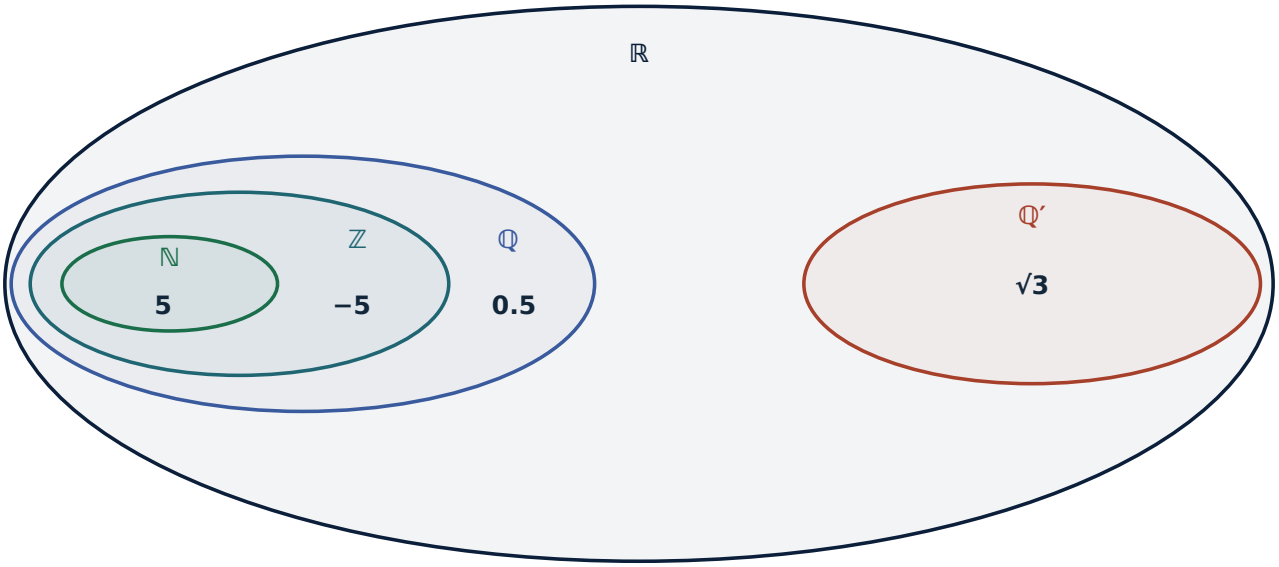
"Smallest family" and "every family" are different questions with different answers. 7 is in four families and its smallest is ℕ; √5 is in two and its smallest is ℚ′. Read which one the question asked for before you write anything down.

Concept 2 · Reading the notation

∈ is an element of · ∉ is not an element of · ⊂ is a subset of

Three symbols carry almost all of the notation. **∈** says a number is in a family, **∉** says it is not, and **⊂** says a whole family sits inside another. The first two join a number to a set; the third joins a set to a set.

WHERE A NUMBER SITS IS WHAT THE NOTATION SAYS



5 ∈ ℕ · −5 ∉ ℕ but −5 ∈ ℤ · 0.5 ∈ ℚ · √3 ∈ ℚ′

Read a membership statement straight off the picture. ∈ asks "is this number inside that oval?" and ∉ asks the opposite. ⊂ is a different question altogether: it asks whether one whole oval sits inside another.

Worked example 2 · Determine whether each statement is true or false

1	$-3 \in \mathbb{N}$ — is -3 a positive integer or zero?	**false**
2	$-3 \in \mathbb{Z}$ — is -3 a whole number?	**true**
3	$\mathbb{Z} \subset \mathbb{Q}$ — is every integer a fraction of two integers?	**true, because n = n/1**
4	$\sqrt{2} \in \mathbb{Q}$ — can $\sqrt{2}$ be written as a fraction?	**false**
5	$0 \in \mathbb{N}$ — does this course count zero as natural?	**true**

$\in$ joins a **number** to a family; $\subset$ joins a **family** to a family. Writing $3 \subset \mathbb{Z}$ or $\mathbb{Z} \in \mathbb{Q}$ is not a wrong answer, it is a sentence that does not parse — and the e-assessment marks notation.

Concept 3 · Why the families nest

Every natural number is an integer, and every integer is rational

$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ is not a convention someone chose — it is forced. Every positive whole number is a whole number; every whole number n is the fraction n/1; every fraction is a point on the number line.

EVERY FAMILY A NUMBER BELONGS TO, NOT JUST THE SMALLEST



The chip colour is the SMALLEST family; the row underneath is every family.

6 belongs to four families and √7 to two, and both statements are complete answers to different questions. Notice that ℝ appears on every chip: nothing in this course escapes it.

Worked example 3 · Classify −7, 0, 3/4, √16, $0.4\overline{16}$ and π into every family

1	Simplify anything that can be simplified, before you classify anything	**√16 = 4**
2	Whole and not negative → all four nesting families	**0 and 4:** ℕ, ℤ, ℚ, ℝ
3	Whole but negative → drop ℕ	**−7:** ℤ, ℚ, ℝ
4	A fraction of integers, or a decimal that stops or repeats → ℚ	**3/4 and $0.4\overline{16}$:** ℚ, ℝ
5	Never ends and never repeats → ℚ′, beside ℚ	**π:** ℚ′, ℝ

Two of these six are traps and both were defused in step 1 or step 4. √16 is not irrational — it is 4. $0.4\overline{16}$ is not irrational — it repeats, so it is 206/495. Every number in the list is in ℝ: that column can never be empty.

Concept 4 · Decimals that stop, and decimals that repeat

Both are rational — and N05 already showed you how to prove it

A decimal that stops is a fraction with a power of ten underneath. A decimal that repeats a block for ever is also a fraction, and N05 gave you the method that produces it. Both kinds are rational, and producing the fraction is the proof.

STOPS, OR REPEATS — EITHER WAY IT IS RATIONAL

0.25 ℚ ℝ	**0.333…** ℚ ℝ	**0.416161…** ℚ ℝ	**√2 = 1.41421…** ℚ′ ℝ

Only the last one never stops AND never repeats.

0.25 stops. 0.333… and 0.416161… never stop, but each repeats a block for ever, so the ×10ⁿ subtraction method from N05 turns them into 1/3 and 206/495. √2 does neither, and that is the whole difference.

Worked example 4 · Show that $0.4\overline{16} \in \mathbb{Q}$

1	Give the number a name and write the tail out in full	**$x = 0.4\overline{16} = 0.4161616\ldots$**
2	One digit sits outside the block, so step past it first	**$10x = 4.161616\ldots$**
3	The block is two digits long, so go 100 times further again	**$1000x = 416.161616\ldots$**
4	Subtract the pair whose tails match — the tails cancel exactly	**$1000x - 10x = 412$, so $990x = 412$**
5	Divide, then cancel the factor of 2	**$x = 412/990 = 206/495$, so $x \in \mathbb{Q}$**

This is the N05 method, unchanged, doing a new job. In N05 it turned a recurring decimal into a fraction; here the fraction **is** the proof of membership, because "rational" means exactly "can be written as a fraction of two integers". Producing the fraction ends the argument.

Concept 5 · What irrational really means

Never ends AND never repeats — and why √16 is not irrational

A number is irrational when its decimal never stops AND never repeats. Both halves matter. The root sign tells you nothing on its own: carry it out first, then classify what you actually have.

THE ROOT SIGN IS AN INSTRUCTION, NOT A LABEL

√16 = 4	√25 = 5	√2	√20
ℕ ℤ ℚ ℝ	ℕ ℤ ℚ ℝ	ℚ′ ℝ	ℚ′ ℝ

√n is rational exactly when n is a square number.

Two of these four roots are natural numbers and two are irrational, and you cannot tell which is which without carrying out the instruction. Simplify first, then classify.

Worked example 5 · Determine whether π ∈ ℚ

1	Write down what would have to be true	**π = a/b for some integers a and b, b ≠ 0**
2	Look at the decimal	**π = 3.14159265358979…**
3	First test: does it stop?	**no — it has no last digit**
4	Second test: does a block of digits repeat for ever?	**no — and it never will**
5	Both tests fail, so no such fraction exists	**π ∉ ℚ; π ∈ ℚ′**

Both tests have to fail. A decimal that stops is rational; a decimal that repeats is rational; only "never stops AND never repeats" is irrational. 22/7 = $3.\overline{142857}$… repeats, so it is rational — it is a good approximation to π, not π.

Worked example 6 · Determine whether √16 and √20 are irrational

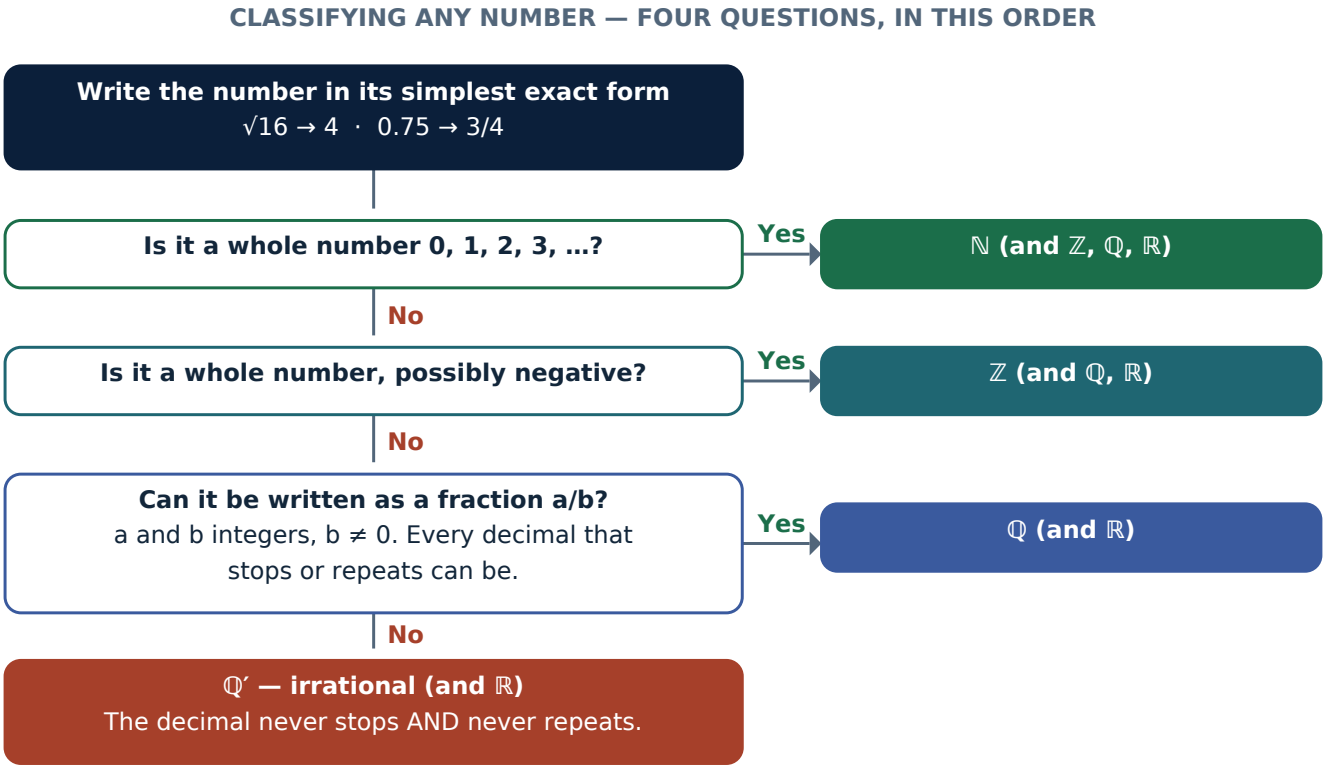
1	Never classify a number by the way it is written. Simplify first	**√16 = 4**
2	4 is whole and not negative	**√16 ∈ ℕ, so it is rational**
3	Now √20. Look for a whole number whose square is 20	**$4^2 = 16$, $5^2 = 25$ — nothing in between**
4	So its decimal never stops and never repeats	**√20 = 4.472135955…**
5	The two roots land in different families	**√16 ∈ ℕ · √20 ∈ ℚ′**

The root sign is an **instruction**, not a label. √n is rational exactly when n is a square number, and irrational otherwise — and the first thing you do with any root is find out which. Of √1 to √10, only three are rational.

Concept 6 · Classifying any number

One routine, and how to answer a "find a number such that" question

Every classification question is the same four questions asked in the same order. The first Yes names the smallest family, and everything larger in the chain comes free. Reaching the bottom without a Yes is what irrational means.



√16 answers Yes at the first question — but only once you have written it as 4.

Work down the left-hand column. The first "Yes" you reach names the smallest family the number belongs to, and everything to the right of it in the chain comes free. Reaching the bottom without a "Yes" is what irrational means.

Worked example 7 · Find x with x ∈ ℚ and x ∉ ℤ, and x with x ∈ ℤ and x ∉ ℕ

1	Read the demand as two tests, not one	**inside the bigger family, outside the smaller**
2	For ℚ but not ℤ, take a fraction that will not cancel	**x = 1/2**
3	Run both tests before you write it down	**1/2 ∈ ℚ** ✓ **· 1/2 ∉ ℤ** ✓
4	For ℤ but not ℕ, go negative — that is all ℕ leaves out	**x = −1**
5	Run both tests again	**−1 ∈ ℤ** ✓ **· −1 ∉ ℕ** ✓

One example is a complete answer to **find a number such that**; one counterexample is a complete answer to **is it always true?**. Neither needs a list. What both need is the two tests written down — the marks are for showing that your number passes them.

The classifying routine

1. Simplify before you classify — √16 is 4. $0.4\overline{16}$ is 206/495. 6/3 is 2. Classify what the number *is*, never what it looks like on the page — this single habit defuses the two most expensive traps in the unit.

2. Ask the whole-number questions first — Is it a whole number and not negative? Then $\mathbb{N}$, and $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ come free. Is it whole but negative? Then $\mathbb{Z}$, and $\mathbb{Q}$, $\mathbb{R}$ come free. The first Yes you reach names the smallest family.

3. Then ask the fraction question — and remember what counts — A fraction of two integers. A decimal that stops. A decimal that repeats a block for ever. All three are the same thing, and all three are $\mathbb{Q}$. Only "never stops AND never repeats" reaches $\mathbb{Q}'$.

✓ **Answer the question that was asked** — "The smallest family" wants one symbol. "Every family" wants a list, and that list always ends with $\mathbb{R}$. "Find a number such that" wants one number that passes both tests, written down with the tests.

The mistakes that cost the most marks

Trap 1 · Assuming that anything with a root sign is irrational

WRONG √16 has a root sign, so $\sqrt{16} \in \mathbb{Q}'$

RIGHT √16 = 4, and $4 \in \mathbb{N}$, so √16 is rational — it is not in $\mathbb{Q}'$ at all

The root sign is an instruction to find a number, not a label saying what kind of number will appear. Carry it out first. √n is rational exactly when n is a square number: √1, √4, √9, √16, √25 are 1, 2, 3, 4, 5, and every one of them is natural. This is the single most expensive slip in the topic, because it turns a one-mark classification into a wrong answer that looks confident.

Trap 2 · Assuming that a long or recurring decimal must be irrational

WRONG 0.4161616… goes on for ever, so it must be irrational

RIGHT the block 16 repeats for ever, so $0.4\overline{16} = 206/495 \in \mathbb{Q}$

Going on for ever is only half the test. An irrational number never ends **and** never repeats; a recurring decimal fails the second half, and the $\times 10^n$ subtraction method from N05 turns it into a fraction every time. If you can find the repeating block, you can find the fraction, and finding the fraction is the proof of rationality. Length proves nothing at all.

Trap 3 · Reading $\mathbb{Q}'$ as "not a number", or thinking $\mathbb{Q} \cup \mathbb{Q}'$ leaves something out

WRONG $\mathbb{Q}'$ is where things go when they are not really numbers, and some real numbers are in neither $\mathbb{Q}$ nor $\mathbb{Q}'$

RIGHT $\mathbb{Q}'$ is the set of real numbers that are not rational; $\mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$ and $\mathbb{Q} \cap \mathbb{Q}' = \varnothing$

√2 and π are numbers: you can locate them on a number line, measure them and calculate with them. The prime on $\mathbb{Q}'$ means "the rest of $\mathbb{R}$", so the two families divide the real numbers between them exactly — nothing is left out and nothing is in both. That is why the diagram draws $\mathbb{Q}'$ beside $\mathbb{Q}$ inside $\mathbb{R}$, and never inside $\mathbb{Q}$.

FOUR MORE ROOTS — TWO NATURAL, TWO IRRATIONAL

√9 = 3	√10	√36 = 6	√40
ℕ ℤ ℚ ℝ	ℚ′ ℝ	ℕ ℤ ℚ ℝ	ℚ′ ℝ

9 and 36 are square numbers; 10 and 40 are not.

Practice · Fluency · 24 questions

P1 **Write down** the symbol used for the set of integers.

P2 **Write down** whether 5 belongs to $\mathbb{N}$.

P3 **Write down** the smallest of the five families that contains -4.

P4 **Write down** what the symbol $\mathbb{Q}'$ stands for.

P11 **Write down** how the symbol $\in$ is read.

P12 **Write down** whether $-3 \in \mathbb{N}$ is true or false.

P13 **Write down** whether $7 \in \mathbb{Q}$ is true or false.

P14 **Write down** how $\mathbb{Z} \subset \mathbb{Q}$ is read.

P21 **Write down** the four nesting families in one chain, using $\subset$.

P22 **Write down** every family that 6 belongs to.

P23 **Write down** every family that -6 belongs to.

P24 **Write down** every family that $2/5$ belongs to.

P31 **Write down** whether 0.25 is rational.

P32 **Write down** 0.4 as a fraction in its lowest terms.

P33 **Write down** $0.\overline{3}$ as a fraction in its lowest terms.

P34 **Write down** whether a decimal that stops is always rational.

P41 **Write down** whether $\sqrt{16}$ is irrational.

P42 **Write down** the value of $\sqrt{49}$ and the smallest family it belongs to.

P43 **Write down** whether π is rational.

P44 **Write down** the two things the decimal of an irrational number does.

P51 **Write down** the smallest family containing 12.

P52 **Write down** the smallest family containing -12.

P53 **Write down** the smallest family containing 0.6.

P54 **Write down** the smallest family containing $\sqrt{11}$.

Practice · Applied · 24 questions

P5 **Write down** the smallest family that contains 3/4.

P6 **Determine** the smallest family that contains 0.

P7 **Determine** the smallest family that contains $\sqrt{2}$.

P8 **Determine** the smallest family that contains -2.5.

P15 **Determine** whether $\sqrt{9} \in \mathbb{N}$ is true or false.

P16 **Determine** whether $0.5 \notin \mathbb{Z}$ is true or false.

P17 **Determine** whether $\mathbb{Q} \subset \mathbb{R}$ is true or false.

P18 **Determine** whether $\pi \in \mathbb{Q}$ is true or false.

P25 **Explain** why every integer is also a rational number.

P26 **Determine** every family that $\sqrt{25}$ belongs to.

P27 **Determine** every family that -1.75 belongs to.

P28 **Determine** every family that $\sqrt{7}$ belongs to.

P35 **Show that** $0.\overline{7} \in \mathbb{Q}$.

P36 **Show that** $0.\overline{27} \in \mathbb{Q}$.

P37 **Determine** the fraction equal to $0.4\overline{16}$.

P38 **Determine** the fraction equal to $0.\overline{12}$.

P45 **Determine** which of $\sqrt{36}$ and $\sqrt{37}$ is irrational.

P46 **Explain** why 22/7 is rational even though it is often used as an approximation for π.

P47 **Determine** whether $\sqrt{0.25}$ is irrational.

P48 **Determine** whether $3 + \sqrt{2}$ is rational or irrational.

P55 **Determine** a value of x for which $x \in \mathbb{Q}$ and $x \notin \mathbb{Z}$.

P56 **Determine** a value of x for which $x \in \mathbb{Z}$ and $x \notin \mathbb{N}$.

P57 **Determine** a value of x for which $x \in \mathbb{R}$ and $x \notin \mathbb{Q}$.

P58 **Determine** whether it is possible to find x with $x \in \mathbb{N}$ and $x \notin \mathbb{R}$.

Practice · Challenge · 12 questions

P9 **Explain** why $\mathbb{R}$ is the largest of the five families, and **write down** the two families whose union is $\mathbb{R}$.

P10 **Determine** one number that is in $\mathbb{Z}$ but not in $\mathbb{N}$, and one number that is in $\mathbb{Q}$ but not in $\mathbb{Z}$.

P19 **Write down** the statement "22/7 is not irrational" in set notation.

P20 **Explain** why $\mathbb{N} \subset \mathbb{Z}$ is true but $\mathbb{Z} \subset \mathbb{N}$ is false.

P29 **Explain** why $\mathbb{Q} \cap \mathbb{Q}' = \varnothing$.

P30 **Determine** whether there is a real number belonging to neither $\mathbb{Q}$ nor $\mathbb{Q}'$.

P39 **Show that** $0.\overline{9} = 1$, and **write down** the smallest family that $0.\overline{9}$ belongs to.

P40 **Determine** the fraction equal to $0.1\overline{6}$, where only the 6 recurs.

P49 **Determine** how many of $\sqrt{1}$, $\sqrt{2}$, $\sqrt{3}$, …, $\sqrt{10}$ are rational.

P50 **Explain** why 3.14159 is rational but π is not.

P59 **List** every family that each of -3, 0, 5/8, $\sqrt{64}$, $0.4\overline{16}$ and π belongs to.

P60 **Determine** whether "a number written with a decimal point is never an integer" is true, and **justify** your answer.

Criterion B · Investigating patterns

Which families survive their own arithmetic?

Add two natural numbers and you get a natural number. Subtract two natural numbers and you might not: $3 - 5 = -2$, which has left $\mathbb{N}$ altogether. A family is called **closed** under an operation when the answer always lands back inside the family, and **not closed** when a single pair of members can throw you out of it.

That word is the whole investigation. Five families, four operations, twenty cells — and one row of the table does something that surprises almost everybody the first time they see it.

Part 1 · structured

Family	+	−	×	÷
$\mathbb{N}$	closed	not closed $3 - 5 = -2 \notin \mathbb{N}$		
$\mathbb{Z}$				
$\mathbb{Q}$				
$\mathbb{Q}'$				
$\mathbb{R}$				

1. **Predict**, before you calculate anything. Write down which of the five families you think are closed under **subtraction**, and which under **division**. Keep the prediction — you will be asked to compare it with what you find.

2. **Complete** the table. In each cell write *closed*, or write *not closed* together with one counterexample: two members of the family whose answer escapes it. The $\mathbb{N}$ row's first two cells are done for you. Ignore

division by zero throughout — it is not an answer that escapes the family, it is not an answer at all.

3. **Identify** what happens as you read down the $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$ rows. Count the closed cells in each row and say in one sentence what growing the family buys you.

4. **Identify** what is unusual about the $\mathbb{Q}'$ row, and **write down** the four counterexamples in full.

5. **Determine** a general rule. Complete this sentence so that it works for any set S and any operation $*$: "S is closed under $*$ when …" Then **determine** the rule for disproving closure — how many counterexamples does it take, and why?

6. **Test and verify** your rule on a family that is not in the table. Try the **even integers** under + and under ×, and the **odd integers** under + and under ×. Predict first, then test, then say whether your rule survived.

7. **Justify** why one counterexample settles "not closed" for ever, but no number of successful examples can settle "closed". Use your $\mathbb{Q}'$ row to explain why a family can look closed for a long time and not be.

Part 2 · open

Now you choose. Take one of these openings or propose your own. You decide which cases to try, how many you need, how to record them, and how you will test whatever you find.

1. **Investigate** whether the product of two irrational numbers is always irrational. Choose your own cases and decide for yourself how many you need. If you find the answer is no, **determine** a rule that describes exactly when the product comes out rational.

2. **Investigate** whether a rational number plus an irrational number is always irrational. This one is true — so **justify** it rather than testing it. Start by supposing the sum were rational and see what that would force $\sqrt{2}$ to be.

3. Choose your own family and **investigate** it: the multiples of 3, the integers from 0 to 10, the numbers of the form $a + b\sqrt{2}$, or one you invent. **Determine** which of the four operations it is closed under, **verify** each answer, and **justify** the pattern you find.

Whichever you choose, take it through the same ladder: **predict**, **identify** the pattern, **determine** a general rule, **test and verify** it on cases you have not already used, and **justify** why it must hold and

where it stops holding.

Marked out of 8 against Criterion B. Filling in the table earns the lower bands; the general rule, the verification and the justification are what reach levels 5–8. Part 2 must show your own choice of method.

Remember, you should communicate in an organized and coherent manner.

Criterion D · Applying maths in real-life contexts

Fencing the round kraal at Tshimologo Farm

Rre Kgosietsile is re-fencing the round cattle kraal at Tshimologo Farm, outside Serowe. He has measured across the middle of the kraal with a 30 m tape and written the measurement down to the nearest 0.1 m. He wants to know how much fencing wire to buy, whether to buy it in rolls or loose, and what the whole job will cost. **The site notes contain more information than you need.**

The site notes

Item	Figure
Kraal, diameter across the middle	24.0 m (30 m tape, nearest 0.1 m)
Gateway already built, width	3.0 m
Strands of wire to run round the kraal	4
Fence posts, one every	3.0 m
Fencing wire, 50 m roll	P489.95
Fencing wire, cut loose, per metre	P11.50
Fence post, each	P78.00
Water reservoir, internal diameter	5.4 m
Water reservoir, depth	2.1 m
Borehole yield	1 800 litres per hour
Cattle on the farm	46
Distance to the hardware shop in Serowe	38 km

The measurements, prices and quantities in these notes were written for this task. They are realistic for a farm outside Serowe, but they are not a real price survey and should not be quoted as one. All prices are in Pula, to 2 decimal places.

What Rre Kgosietsile tells you

• The kraal is a circle, and the measurement across the middle is its **diameter**.

• The gateway is already built. No wire runs across it.

• Wire is sold either in whole 50 m rolls or cut loose by the metre — he will not buy part of a roll.

• Posts go all the way round, including past the gateway.

• He collects the materials himself in the farm bakkie.

Identify which rows of the site notes you actually need in order to answer the questions below, and **explain** why each of the others is not relevant. There is more here than you need. *(Strand i — identifying the relevant elements)*

Select your strategy and **explain** it. Say which formula gives the distance round the kraal, and **explain** the decision you are going to make about π: at what point in the working will you turn it into a decimal, and why not earlier? *(Strand ii — selecting appropriate strategies)*

Apply it. (a) **Calculate** the distance round the kraal, first as an exact value in terms of π and then to 3 significant figures. (b) The gateway is already built and needs no wire. **Calculate** the total length of wire for 4 strands round the rest. (c) **Determine** how many whole 50 m rolls that needs and what they cost, and **compare** it with buying the same wire loose by the metre. (d) **Determine** the number of posts and the total bill for wire and posts. *(Strand iii — applying the strategy)*

Explain how accurate your answers are. Three things to deal with. First, **explain** which of your figures is exact and which is not, and why the distance round the kraal can never be written exactly as a decimal. Second, the diameter was measured to the nearest 0.1 m: **calculate** the largest and smallest wire lengths that measurement allows, and **explain** whether the difference changes what you would buy. Third, **explain** why the number of rolls is rounded *up* rather than to the nearest whole number, and why that is not a breach of the round-don't-truncate rule. *(Strand iv — accuracy of the solution)*

Discuss whether your answers make sense on the farm. Is an answer left in terms of π more useful to a builder buying materials than a decimal, or less? **Discuss** what the spare wire on the last roll is worth, whether 4 strands and a post every 3 m is a sensible specification for cattle, and one cost the site notes do not mention at all. *(Strand v — does it make sense in context)*

Submit: your exact and rounded lengths, your roll-versus-loose comparison with both costs, your total bill, and written answers to strands i, iv and v. **Marked out of 8** against Criterion D. This is the Standard band, so strands iv and v ask you to **explain** and **discuss**, not merely to describe.

Where you go next

N13 How Far From Zero is next. It takes the number line you have just populated and asks one question about it — how far is this number from 0? — which is what the absolute value |x| measures. **Working with surds lives in Algebra Unit 19.** This unit needed only to know that √2 is irrational; simplifying √45, adding like surds and rationalising a denominator are taught once, there.

Answers

How the P-numbers run. The 60 practice items are numbered P1 to P60 in the order they appear in the lesson — ten per concept, and inside each concept four Fluency, then four Applied, then two Challenge. The same P-number identifies the item in the lesson, in this key and in the teacher pack, so the numbers inside each tier below step in blocks of ten rather than running consecutively: Fluency is P1–P4, P11–P14, P21–P24 and so on; Applied is P5–P8, P15–P18, …; Challenge is P9–P10, P19–P20, …

Fluency · F-tier (24)

P1	$\mathbb{Z}$
P2	**yes — $5 \in \mathbb{N}$**
P3	$\mathbb{Z}$
P4	**the set of irrational numbers**
P11	**is an element of**
P12	**false**
P13	**true**
P14	**$\mathbb{Z}$ is a subset of $\mathbb{Q}$**
P21	$\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$
P22	$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$
P23	$\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$
P24	$\mathbb{Q}$, $\mathbb{R}$
P31	**yes**
P32	**2/5**
P33	**1/3**
P34	**yes**
P41	**no — $\sqrt{16} = 4$**

P42	**7, and the smallest family is ℕ**
P43	**no**
P44	**it never ends and it never repeats**
P51	ℕ
P52	ℤ
P53	ℚ
P54	ℚ′

Applied — fully worked (24)

P5 ℚ

3/4 is not a whole number, so ℕ and ℤ are both too small. It is a fraction of two integers with a non-zero denominator, which is exactly the definition of a rational number, so the smallest family is ℚ.

P6 ℕ

ℕ in this course is the set of positive integers *and zero*, so 0 is already inside the smallest family. It is also in ℤ, ℚ and ℝ — but the question asks for the smallest.

P7 ℚ′

√2 = 1.414213562… The decimal never stops and no block of digits ever repeats, so it cannot be written as a fraction of two integers. That puts it in ℚ′, and ℚ′ is not inside ℚ — it sits beside it.

P8 ℚ

−2.5 is negative, so it is not in ℕ, and it is not whole, so it is not in ℤ. It stops after one decimal place, so it is a fraction: −2.5 = −5/2 ∈ ℚ.

P15 true

Simplify before you classify: √9 = 3. Three is a whole number and it is not negative, so 3 ∈ ℕ and the statement is true. The root sign is not part of the number — it is an instruction.

P16 true — 0.5 ∉ ℤ

∉ means "is not an element of", so the statement claims 0.5 is not an integer. It is not: 0.5 = 1/2 sits halfway between 0 and 1. The statement is

therefore true.

P17 true

$\subset$ asks whether *every* member of the first set is a member of the second. Every rational number is a real number, so yes. Note that $\subset$ joins two *sets* while $\in$ joins a *number* to a set — $3 \subset \mathbb{Z}$ is not a statement, it is a spelling mistake.

P18 false

$\pi = 3.141592653\ldots$ and its decimal neither terminates nor settles into a repeating block, so it is not a fraction of two integers. $\pi \notin \mathbb{Q}$; $\pi \in \mathbb{Q}'$.

P25 because any integer n can be written as the fraction n/1

The definition of a rational number is "a number that *can* be written as a/b with a and b integers and $b \neq 0$". For an integer n, take a = n and b = 1: $7 = 7/1$ and $-6 = -6/1$. Nothing says the fraction has to look like a fraction.

P26 $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$

$\sqrt{25} = 5$, and 5 is a whole number that is not negative. So it is in $\mathbb{N}$, and therefore automatically in $\mathbb{Z}$, $\mathbb{Q}$ and $\mathbb{R}$ as well. The root sign changed nothing except the amount of work.

P27 $\mathbb{Q}$, $\mathbb{R}$

-1.75 stops after two decimal places, so it is a fraction: $-1.75 = -175/100 = -7/4$. It is not whole, so it is not in $\mathbb{Z}$ and not in $\mathbb{N}$.

P28 $\mathbb{Q}'$, $\mathbb{R}$

7 is not a square number: $2^2 = 4$ and $3^2 = 9$, and there is no whole number in between. So $\sqrt{7} = 2.645751311\ldots$ never terminates and never repeats, which puts it in $\mathbb{Q}'$. Every number in this course is real, so it is in $\mathbb{R}$ too — but it is in neither $\mathbb{N}$, $\mathbb{Z}$ nor $\mathbb{Q}$.

P35 $0.\overline{7} = 7/9$

Let $x = 0.\overline{7} = 0.777\ldots$ The block is one digit long, so multiply by 10: $10x = 7.777\ldots$ Subtract: $10x - x = 7$, so $9x = 7$ and $x = 7/9$. That is a fraction of two integers, so $0.\overline{7} \in \mathbb{Q}$.

P36 $0.\overline{27} = 3/11$

The block is two digits long, so multiply by 100, not 10. $x = 0.272727\ldots$ and $100x = 27.272727\ldots$ Subtract: $99x = 27$, so $x = 27/99 = 3/11$.

P37 206/495

One digit sits outside the block, so step past it first: $x = 0.4161616\ldots$ and $10x = 4.161616\ldots$ The block is two digits long, so also form $1000x =$

416.161616… Subtract the pair whose tails match: 1000x − 10x = 412, so 990x = 412 and x = 412/990 = 206/495.

P38 4/33

x = 0.121212…, and the block is two digits, so 100x = 12.121212… Subtract: 99x = 12, so x = 12/99 = 4/33.

P45 √37 is the irrational one

36 is a square number, so √36 = 6 ∈ ℕ. 37 is not a square number: 6^2 = 36 and 7^2 = 49. So √37 = 6.082762530… never ends and never repeats, and it is the irrational one. Two roots that look alike can land in completely different families.

P46 22/7 is a fraction of two integers, so it is rational

Being close to an irrational number does not make a number irrational. 22/7 = 3.142857142857… — the block 142857 repeats for ever, which is the signature of a rational number. π = 3.14159265… never repeats. The two numbers agree to two decimal places and then part company for good.

P47 no — √0.25 = 0.5, which is rational

0.25 = 1/4, and the square root of 1/4 is 1/2. So √0.25 = 0.5 ∈ ℚ. A root does not have to be a root of a whole number to come out exact.

P48 irrational

Suppose 3 + √2 were rational, say equal to r. Then √2 = r − 3, and a rational number minus a rational number is rational — which would make √2 rational. It is not, so 3 + √2 cannot be rational either. Adding a rational number to an irrational one always leaves it irrational.

P55 any non-whole fraction, for example x = 1/2

The demand is two tests, not one: in the bigger family, out of the smaller. 1/2 is a fraction of two integers, so 1/2 ∈ ℚ ✓; and it is not a whole number, so 1/2 ∉ ℤ ✓. Any answer that passes both tests earns full marks — 0.7, −3/4 and 2.5 all work.

P56 any negative integer, for example x = −1

ℕ holds the positive integers and zero, so the integers it leaves out are exactly the negative ones. −1 ∈ ℤ ✓ and −1 ∉ ℕ ✓. Answering 0 fails the second test, because 0 *is* in ℕ.

P57 any irrational number, for example x = √2

A real number that is not rational is by definition irrational, so the question is asking for a member of ℚ′. √2, √3, π and 1 + √5 all do the job. √4 does

not: it is 2.

P58 no — $\mathbb{N} \subset \mathbb{R}$

$\mathbb{N}$ sits inside $\mathbb{Z}$, which sits inside $\mathbb{Q}$, which sits inside $\mathbb{R}$. So every natural number is real, and there is nothing to find. A question can ask for a counterexample that does not exist, and saying so — with the reason — is the full answer.

Challenge — fully worked (12)

P9 $\mathbb{Q}$ and $\mathbb{Q}'$; $\mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$

Every number in this course is real. A real number either can be written as a fraction of two integers or it cannot: there is no third possibility. The ones that can are $\mathbb{Q}$, the ones that cannot are $\mathbb{Q}'$, so together they are the whole of $\mathbb{R}$. Nothing is left over and nothing is counted twice.

P10 $-3 \in \mathbb{Z}$ but $-3 \notin \mathbb{N}$; $1/2 \in \mathbb{Q}$ but $1/2 \notin \mathbb{Z}$

Any negative whole number does the first job, because $\mathbb{N}$ stops at 0. Any fraction that does not cancel to a whole number does the second. Both answers show that the nesting is *strict*: each family is genuinely bigger than the one inside it.

P19 $22/7 \notin \mathbb{Q}'$

"Irrational" is the family $\mathbb{Q}'$, and "is not" is $\notin$. You could equally write $22/7 \in \mathbb{Q}$, which says the same thing, because a real number that is not irrational must be rational.

P20 $\mathbb{N} \subset \mathbb{Z}$ is true; $\mathbb{Z} \subset \mathbb{N}$ is false because $-1 \in \mathbb{Z}$ but $-1 \notin \mathbb{N}$

Every member of $\mathbb{N}$ is a whole number that is not negative, so every one of them is an integer — the first statement holds with no exceptions. The second statement needs every integer to be in $\mathbb{N}$, and -1 is not. One counterexample is enough to destroy a $\subset$ claim; no number of examples can prove one.

P29 no number can be both rational and irrational, so the two families share nothing

$\mathbb{Q}'$ is *defined* as the real numbers that are not in $\mathbb{Q}$. A number that was in both would have to be a fraction of two integers and not a fraction of two integers at the same time. So the overlap is empty — which is why the $\mathbb{Q}'$ oval in the diagram sits beside $\mathbb{Q}$ and not inside it.

P30 no — $\mathbb{Q} \cup \mathbb{Q}' = \mathbb{R}$

Take any real number and ask one question: can it be written as a fraction of two integers? If yes it is in $\mathbb{Q}$; if no it is in $\mathbb{Q}'$. There is no third answer, so nothing can escape both. Together with P29 this says the two families *partition* $\mathbb{R}$: no gaps, no overlap.

P39 $0.\overline{9} = 1$, so the smallest family is $\mathbb{N}$

x = 0.999..., so 10x = 9.999... Subtract: 9x = 9, so x = 1 exactly. This is not a rounding error and it is not "nearly 1" — the two expressions name the same number. Since that number is 1, the smallest family containing it is $\mathbb{N}$.

P40 1/6

One digit is outside the block and the block is one digit long. x = 0.1666..., 10x = 1.666... and 100x = 16.666... Subtract the matching tails: 100x − 10x = 15, so 90x = 15 and x = 15/90 = 1/6. Moving the dot changes the number: $0.\overline{16}$ is 16/99, not 1/6.

P49 3 — √1, √4 and √9

√n is rational exactly when n is a square number. Between 1 and 10 the square numbers are 1, 4 and 9, giving √1 = 1, √4 = 2 and √9 = 3. The other seven roots are irrational. Notice how few of them are rational — which is why the trap in this unit is worth so many marks.

P50 3.14159 stops, so it equals 314159/100000; π never stops and never repeats

3.14159 is a terminating decimal, and every terminating decimal is a fraction with a power of ten underneath. π has no last digit and no repeating block, so no such fraction exists. The two numbers agree to five decimal places, and agreeing to any number of decimal places is not the same as being equal.

P59 −3: $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ · 0: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ · 5/8: $\mathbb{Q}$, $\mathbb{R}$ · √64: $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$ · $0.4\overline{16}$: $\mathbb{Q}$, $\mathbb{R}$ · π: $\mathbb{Q}'$, $\mathbb{R}$

Simplify first: √64 = 8, so it behaves exactly like any other natural number. $0.4\overline{16}$ recurs, so it is the fraction 206/495 and therefore rational. −3 is whole but negative. 5/8 is a fraction that will not cancel to a whole number. π is the only member of $\mathbb{Q}'$ in the list. Every one of the six is in $\mathbb{R}$ — that column is never empty.

P60 false — 4.0 = 4, and 4 ∈ $\mathbb{Z}$

The claim is about how a number is *written*, not about what it *is*. 4.0 has a decimal point and is the integer 4; so does 12.00. Even $0.\overline{9}$ is the integer 1. One counterexample settles it, and the counterexample has to be a

number, not an argument.